The Apparent Irreversibility of Particle Creation: A Study of Time Scales and of the Mechanisms Responsible for Entropy Production in Quantum Field Theory

by

Jochen Rau

Department of Physics
Duke University

Date:______________________________

Approved:

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Berndt Müller, Supervisor

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Dissertation submitted in partial fulfillment of
the requirements for the degree of Doctor
of Philosophy in the Department of
Physics in the Graduate School
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1998

ABSTRACT

(Theoretical Physics)

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Abstract

In the presence of strong gravitational, electromagnetic or other gauge fields, particle-antiparticle pairs are created out of the vacuum. Creation processes of this type are responsible for, e.g., hadron production in heavy ion collisions or the radiation of black holes. They lead to an increase in entropy, thus contributing to the thermalization of the system under consideration. This suggests that particle creation in strong fields is an irreversible process. Key issues to be addressed are: (i) under which conditions particle creation is indeed irreversible, and how this can be reconciled with the time-reversal invariance of the underlying microscopic dynamics; (ii) if — and if so, how — particle creation can be described within the framework of a theory of nonequilibrium processes; (iii) how the associated entropy is defined; and (iv) how particle creation can be incorporated into a kinetic equation that also accounts for subsequent acceleration and collisions.

These issues will be studied by means of the projection method. After a comprehensive introduction to that method, it will be applied to a simple model from quantum electrodynamics which incorporates acceleration, collisions, and pair creation due to the Schwinger mechanism. For this model, I will obtain (1) the complete set of time scales, which furnishes a precise mathematical criterion for the irreversibility of particle creation; (2) the associated relevant entropy to which the H-theorem applies; and (3) a generalisation of the quantum Boltzmann equation which includes a source term derived from first principles.

Meinen Eltern

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Chapter 1

Introduction

1.1 Motivation

The motivation for this investigation into the apparent irreversibility of particle creation is taken from two seemingly distinct areas of modern research, relativistic heavy ion physics and black hole thermodynamics. In both these areas, one encounters physical systems in which sufficiently strong fields may exist for particles to be created out of the vacuum.

Relativistic Heavy Ion Physics

Heavy ion collisions display a high degree of irreversibility: whereas the incoming particles enter the collision zone in an almost pure state, particles emerging from the reaction can very well be described by a thermal distribution (fig. 1.1). Efficient thermalisation mechanisms must be at work. Some of these are well understood, such as multiple scattering of the microscopic constituents of the colliding ions, or chemical reactions. Recently it has also been suggested [1, 2] that classical non-Abelian Yang-Mills theories, and hence chromodynamics, are deterministically chaotic, thus leading to an effective 'self-thermalisation' of the gauge fields.

None of these mechanisms, however, can account for a process that is unique to *quantum* field theory and thus has no classical analogue: particle creation in strong fields. In the course of a heavy ion collision regions of very large field strengths (electromagnetic or chromoelectric) may develop, which subsequently decay by emitting (color-)charged particle-antiparticle pairs. Such pair creation in strong chromoelectric fields gives rise to the fragmentation of chromoelectric flux tubes ('strings'), a mechanism that is frequently invoked to model hadron production in high energy reactions [3, 4, 5, 6]. Particles are produced with a certain momentum distribution, associated with which is clearly a non-zero entropy. Hence particle creation in strong fields certainly does contribute to the thermalisation of the total system. But is particle creation, therefore, an irreversible process? And if so, is it not puzzling that entropy, just like particles, should be produced out of the vacuum?

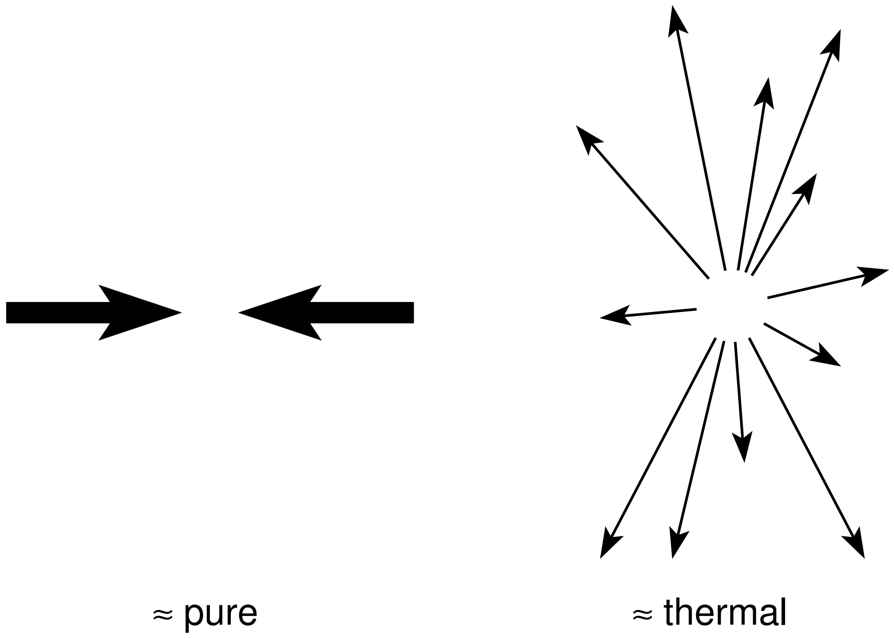

Figure 1.1: Heavy ion collisions are highly irreversible.

Black Hole Thermodynamics

Particle creation occurs not only in gauge fields, but also in gravitational fields. For this reason black holes are not totally black: particle-antiparticle pairs are produced near the horizon and give rise to the so-called Hawking radiation [7, 8, 9, 10]. This radiation is thermal.* Hence while energy (in the form of matter) enters the black hole in an almost pure state, radiation is emitted completely thermalised. Again, the question arises how this thermalisation and the associated entropy production are achieved; and again, particle creation in strong fields is an important — if not the only — contributor. In order to understand the thermodynamics of black holes, therefore, it is essential to shed light on this apparent irreversibility of particle creation.

1.2 Conceptual and Practical Issues

Conceptual Issues

Some basic questions immediately arise: Is particle creation truly irreversible? Or is the irreversibility only apparent (as the title indicates)? If so, in what sense? What is the associated entropy? In which sense is it being produced? How does the (apparent) irreversibility manifest itself in the mathematical formalism? In particular, how can it be reconciled with the time-reversal invariance of the underlying microscopic dynamics?†

As I will discuss in chapter 2, rigorous answers to these questions rest on such concepts as 'relevant observables,' 'levels of description' and 'coarse graining.' Moreover, I will emphasize that irreversibility is closely linked to the existence of various clearly separated time scales in the system. In order to be able to apply a generalised H-theorem, the time scales on which the relevant observables evolve must be much larger than the so-called 'memory time' which accounts for residual effects of the irrelevant degrees of freedom; only then may the evolution of the relevant observables be described, to a very good approximation, by a Markovian equation of motion. Otherwise there may be sizeable memory effects.

A clear separation of time scales may not be taken for granted, and indeed studies have shown [14, 15] that in the usual Boltzmann description of nuclear collisions, for example, memory effects may be quite large — in which case the Boltzmann description is no longer adequate, and additional 'hidden variables' have to be included in the level of description in order to restore its Markovian property. Before applying any coarse-grained description such as the Boltzmann description, it is therefore necessary to perform a very careful analysis of time

*It can be assigned a well-defined temperature; cf. also [11, 12, 13].

†Fundamental T-violations on the microscopic level will not be considered here.

scales. Such an analysis must also precede any attempt to understand the apparent irreversibility of particle creation.

In a broader context, particle creation is just one of many physical processes for which the question of the applicability of kinetic equations arises. It may seem surprising that indeed so often it is possible to describe physical processes — albeit quantum mechanical on the microscopic level — with essentially classical kinetic equations. Certainly within the realm of everyday experience we are perfectly justified in applying classical equations of motion. How is this possible, given that the underlying microscopic dynamics is most certainly governed by the laws of quantum mechanics? This phenomenon of 'emerging classicality' or 'decoherence,' as it is now known, has recently received renewed attention. For open systems, decoherence can usually be linked to the interaction with a surrounding reservoir (as is constituted by, e.g., the cosmic microwave background). The argument becomes more subtle for isolated systems; in particular, for the universe as a whole, which is really the only truly isolated system. Two approaches have been suggested to deal with this issue, one introduced by Zurek and others [16, 17, 18, 19, 20], and one originated by Gell-Mann and Hartle [21, 22]. Zurek et al work in the conventional Hilbert space framework and base their argument on the concept of an 'environment' (which need not be spatially separated from the system under consideration, but stands rather generically for the set of all irrelevant degrees of freedom), leading to the notions of a preferred 'pointer basis' and of so-called 'Schmidt paths.' In contrast, Gell-Mann and Hartle employ the path integral formalism, the central mathematical object being the so-called 'decoherence functional.' In this thesis I do not intend to enter the current debate on decoherence (see, e.g., [23, 24]); however, many of the concepts that I will use (such as relevant observables, levels of description, coarse graining, time scales, etc.) are matched by similar concepts used — sometimes under different names — in the modern literature on decoherence.

Practical Issues

Some of the conceptual issues raised above immediately translate into practical tasks, such as (1) finding an appropriate (coarse-grained) level of description; (2) determining the memory time as well as the time scales for the evolution of the relevant observables; and thus (3) establishing a criterion for the validity of a (Markovian) kinetic description of particle creation. Once the Markovian property is established, an H-theorem and with it the apparent irreversibility of particle creation immediately follow.

Moreover, the question arises how particle creation can be incorporated into a more general kinetic theory that accounts not only for particle creation itself, but also for collisions and for acceleration of the particles in the external field. This will lead to a generalisation of the usual (quantum) Boltzmann equation, which has to be modified by an additional source term. In the literature (e.g., [25, 26, 27]), an expression for this source term is often hypothesized without any rigorous justification. In particular, it is not known how the longitudinal momenta of the produced particles are distributed. To derive the source term from first principles, therefore, is an important major task.

On the coarse-grained level of description one defines the associated 'relevant entropy,' whose increase gives rise to the perception of irreversibility. In the framework of the extended quantum Boltzmann equation, both collisions and particle creation contribute to this increase. Several questions arise (fig. 1.2): At which rate is (relevant) entropy being produced due to particle creation alone? Does this already lead to complete equilibration? (And if so, what is the equilibration time?) Or does particle creation only 'pre-equilibrate' the system? (And if so, how far does the system remain removed from equilibrium?) How does particle creation interplay with the other predominant thermalisation mechanism, namely collisions? What is their relative importance? Are there, for example, various temporal regimes in which either of these thermalisation mechanisms dominates?

1.3 Model

I will address the above issues in the framework of quantum electrodynamics (QED). More specifically, I will consider the creation of electron-positron pairs in a strong homogeneous, time-independent electric field, a process which is also referred to as the Schwinger mechanism [28, 29, 30, 31]. This model is certainly an idealisation, and may differ in some respects from the realistic configurations which motivate this study. Not only is the field of a different type (electromagnetic rather than chromoelectric or gravitational), but also the assumptions of time-independence and homogeneity, for example, may be oversimplifications. Yet this model, despite its relative simplicity, already exhibits all the essential features that are of interest to us: existence of various time scales; apparent irreversibility; and, as a result, the generation of (relevant) entropy. As a first step towards understanding the role of particle creation in the equilibration of nuclear or cosmological systems, therefore, the QED model serves its purpose well.

There is, in fact, a closely related phenomenon that has been known for almost 60 years: the 'breakdown of a solid dielectric' (insulator) due to particle-hole creation in the presence of a strong electric field [32, 33, 34, 35, 36, 37]. The mathematical result for the total creation rate is almost identical with the famous Schwinger formula. If anything, *this* may be regarded as the generic, most intuitive model for pair creation. (The effect of subsequent collisions can also be studied, by adding impurities to the insulator.) Although this model from solid state physics may sometimes aid the intuition, I will not pursue the analogy any further.

I should caution the reader that the QED model may differ from its cosmo-

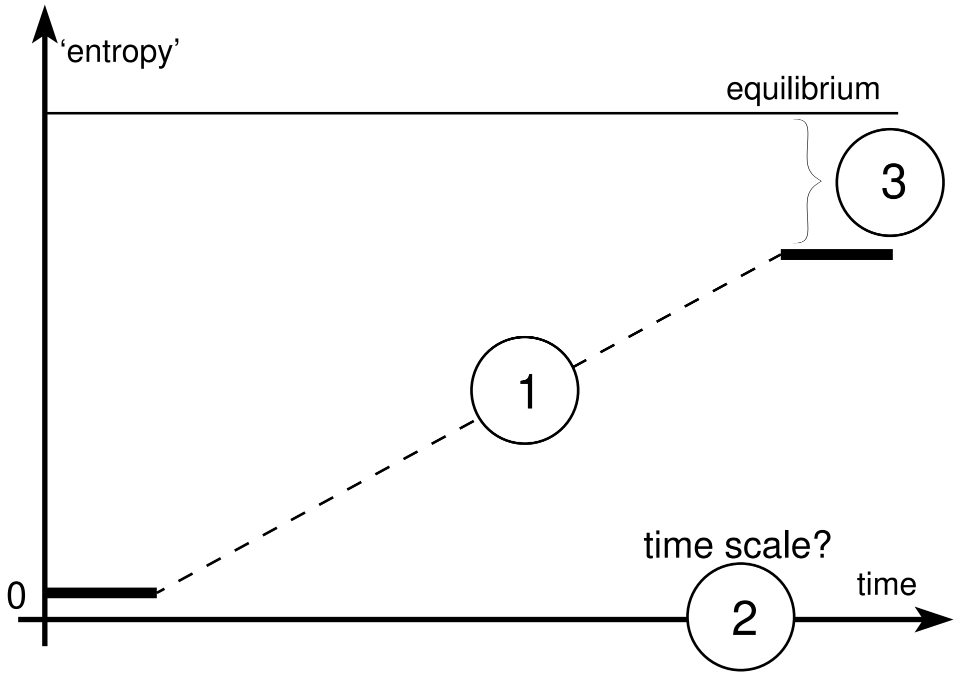

Figure 1.2: Some questions of practical importance: (1) At which rate is (relevant) entropy being produced due to particle creation? (2) If so, what is the equilibration time? (3) Or does particle creation only 'pre-equilibrate' the system? If so, how far does the system remain removed from equilibrium?

logical progenitor in one *conceptual* aspect: I will argue that in the QED case, the apparent irreversibility stems from the fact that information about the system is being transferred into practically inaccessible degrees of freedom such as complicated correlations and rapidly oscillating phases. The information is lost only in practice, not in principle. In the case of Hawking radiation, however, it may well be that some of the information about the system cannot even be retrieved in principle. This is related to the existence of an event horizon: an external observer cannot know what is happening inside the black hole, and thus cannot determine the complete particle-antiparticle correlations or coherence — not even in principle. Whether the resulting information loss is truly final, or whether the information may (theoretically) be retrieved once the black hole has evaporated through continued radiation, is the subject of current scientific debate [38, 39, 40, 41, 42, 43, 44].

One might argue that an equally inevitable information loss may occur in the case of QCD. As a result of confinement, the color configuration inside a produced hadron is unobservable, and therefore any information about the color configuration must be irretrievably lost. However, confinement can always be broken up if 'only' sufficient energy is available; in fact, in heavy ion collisions at extremely high energies, nuclear matter is expected to reach such a deconfined plasma phase (the quark-gluon plasma [45]). Hence, strictly speaking, the loss of information about the color configuration is imposed only by practical, not theoretical limitations.

For the problems I am considering, however, the above discussion is rather academic. Whether the information loss results from practical limitations or occurs in principle has no bearing on the various practical issues that have been raised; this particular conceptual question will not affect the results of my study.

1.4 Mathematical Approach

In the literature, particle creation is often described mathematically by means of a Wigner function formulation [46, 47, 48, 49, 25, 26]. Wigner functions are the quantum mechanical analogue of a single-particle phase space distribution. In the Wigner description, all particle correlations have been removed, and thus only single-particle information is retained. Wigner functions evolve according to a Vlasov-like equation of motion, which is why they seem an appropriate starting point for any attempt to formulate a kinetic theory involving particle creation. In contrast to their classical counterpart, however, Wigner functions do not lend themselves to a probabilistic interpretation; their values may be negative, even complex. This lack of a probabilistic interpretation implies that there is no obvious way of defining an associated (relevant) entropy, either. Moreover, it has not been considered how, in the presence of pair creation in a strong electric field, a collision term can be incorporated into the equation of motion.

I will take a different approach. Since in general the only experimentally measured quantities are the momenta of the outgoing particles, these constitute the only relevant observables. This renders it appropriate to choose a mathematical description that carries information about the single-particle momenta *only*. Rather than with a phase-space distribution, therefore, the system will be described with a momentum space distribution.

In order to obtain the equation of motion for the momentum space distribution, I will apply a very powerful and broadly applicable mathematical tool: the projection method. It is formulated in the language of density matrices, and hence allows for a direct probabilistic interpretation. Crucial concepts such as the level of description, (relevant) entropy, memory time etc., can be easily defined, making the projection method especially suited for the issues surrounding the apparent irreversibility of particle creation. Finally, mathematical manipulations and approximations within this formalism can be easily interpreted physically, providing useful guidance during complicated calculations and rendering their meaning more transparent.

1.5 Outline

Chapters 2 and 3 will be devoted to a discussion of conceptual issues and a presentation of the general mathematical framework used to describe irreversible processes. I will take a closer look at the well-known 'irreversibility paradox,' and show how it can be resolved by introducing the notions of relevant observables, coarse graining, levels of description and associated relevant entropy. In particular, I will emphasize the intimate connection between irreversibility and the existence of strongly separated time scales in the system. I will show how then the Markovian limit immediately leads to an H-theorem. Most of chapter 3 will be concerned with the main mathematical tool for my study, the projection method. After a general introduction to that formalism, I will survey special cases and common approximations, some of which will be employed later. To illustrate the various techniques and approximations, I will discuss the decay of a single resonance weakly coupled to a continuum as a simple, yet instructive, example.

Later in this thesis, I will concentrate on two basic mechanisms that may lead to entropy production: (i) collisions and (ii) particle creation in strong fields. They will be investigated in detail in chapters 4 and 5, respectively.

In chapter 4, I will study how a quantum many-body system evolves under the influence of binary collisions. Using the projection method, I will be able to derive a generalisation of the quantum Boltzmann equation (including both the acceleration and collision terms) which can account for finite memory effects. I will perform a thorough time scale analysis and show under which conditions a Markovian limit is justified. Whenever it is, the general results from chapter 2 immediately imply an H-theorem and thus the production of relevant entropy. Apart from illustrating the role that collisions play in the equilibration of a many-particle system, chapter 4 will also serve a more general purpose: it will demonstrate how the projection method can be used to obtain kinetic equations from first principles.

Chapter 5, finally, will address the central topic of this thesis: particle creation and its potential contribution to the production of (relevant) entropy. First it will be necessary to gain an understanding of the particle creation process on the microscopic level. For this purpose I will present a useful analogy that links pair creation in a strong electric field to the flipping of a spin $1/2$ in a time-dependent magnetic field. Next I will show how particle creation can be incorporated into a kinetic description, leading to the extension of the quantum Boltzmann equation by an additional source term. This source term will be derived from first principles, and its consistency with the Schwinger formula will be checked. The source term, too, will then be subjected to a time scale analysis, establishing a criterion for the validity of the Markovian limit. Again, the Markovian limit — if justified — will immediately imply an H-theorem. We will then be in a position to answer the original questions: Is particle creation irreversible? If so, in what sense? Why is the irreversibility only apparent? Where does the information go?

In chapter 6, the main findings of this study will be summarized. I will review the time scales, relevant entropy, and kinetic equation, of a system experiencing both collisions and particle creation. I will also look at future tasks, which include solving the complete kinetic equation — with source term — by numerical means as well as various mathematical extensions of the model considered here. Finally, the results of this thesis will lead us to a re-examination of the original agenda and a discussion of the lessons we have learnt about the origin of the apparent irreversibility of particle creation.

There are three appendices. While two of them (A and C) deal with rather technical issues, appendix B contains a condensed review of some of the basic results of (equilibrium) quantum statistical mechanics.

Chapter 2

Irreversibility and Time Scales

Experiments in high energy physics, such as collisions of elementary particles or heavy ions, confront us with a startling paradox: While the initial state of the system (before the collision) is almost pure, particles emerging from the reaction can very well be described by a thermal distribution. This thermalization is accompanied by a large apparent increase in entropy, indicating that the process must be highly irreversible; yet at the same time the entire reaction is governed by the laws of quantum mechanics which are invariant under time reversal.*

This is just a special case of the so-called *irreversibility paradox* that has plagued physicists since the early days of thermodynamics [50, 51]. Indeed, the apparent contradiction between microscopic reversibility and macroscopic irreversibility has long been a source of confusion and puzzlement. To understand the paradox, let us briefly recall some basic facts: (i) The state of a quantum system at time t is given by a statistical operator (density matrix) $\rho(t)$ with the properties

$$\rho^\dagger = \rho \quad , \quad \rho \geq 0 \quad , \quad \operatorname{tr} \rho = 1 \quad ; \tag{2.1}$$

(ii) its v. Neumann entropy is defined as

$$S_{\text{v.N.}} := -k \operatorname{tr}(\rho \ln \rho) \quad , \tag{2.2}$$

k being the Boltzmann constant; (iii) if the system is isolated, the statistical operator evolves in the Schrödinger picture according to the Liouville-v. Neumann equation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[H(t), \rho(t)] \quad , \tag{2.3}$$

where $H(t)$ is the (possibly time-dependent) Hamiltonian; and (iv) the Liouville-v. Neumann equation implies that the v. Neumann entropy remains constant in time. — But this obviously contradicts our observation that the entropy increases! (Note that although we formulated the irreversibility paradox in the language of quantum mechanics, it is not a quantum phenomenon; it appears just as well in classical statistical mechanics.)

How can this apparent paradox be resolved? Clearly two possible solutions present themselves:

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*Violations of microscopic time reversal symmetry due to weak interaction will not be considered here.

1. The system is not isolated but rather subject to random perturbations. Its Hamiltonian is not exactly known; instead, various Hamiltonians H_α apply with respective probabilities p_α. Consequently the time evolution can no longer be described by the Liouville-v. Neumann equation. Rather we find that after a random perturbation
$$\rho \to \sum_\alpha p_\alpha U_\alpha \rho U_\alpha^+ \quad , \tag{2.4}$$
U_α being the unitary evolution operator corresponding to H_α. It can be shown that any such random perturbation can only increase the v. Neumann entropy [53, 51].

2. The v. Neumann entropy is not the appropriate quantity for the description of irreversibility. It differs from the 'relevant' entropy whose increase one observes: $S_{\text{v.N.}} \neq S_{\text{rel}}$. How this relevant entropy is defined must be the subject of further investigation. Its definition involves a procedure commonly referred to as 'coarse graining'; but this term merely captures the basic concept of eliminating 'irrelevant' degrees of freedom. To be practically useful, the coarse graining procedure has to be carefully defined.

The first of these two solutions can in fact always be reduced to the second one. It can be shown that any random perturbations of the form (2.4) can be realised through the interaction of the considered system S with some external system (reservoir) R [54, 55]. The combined system $S + R$ is, of course, isolated and evolves according to the Liouville-v. Neumann equation. Therefore a system subject to random perturbations can always be regarded as a subsystem of a larger isolated system. But while random perturbations could explain irreversibility for the smaller system S, for the larger isolated system $S + R$ the irreversibility paradox re-appears! Inevitably we must then resort to our second solution to the irreversibility paradox: The combined system's v. Neumann entropy $S_{\text{v.N.}}^{s+r}$ is not the relevant entropy to be considered.

Let us henceforth focus on isolated systems and thus on the second solution to the irreversibility paradox. One well-known example is a dilute classical gas whose time evolution is described by the Boltzmann equation. Strictly speaking, the state of such a system of N indistinguishable particles is given by a symmetric probability density $W(\pi_1 \ldots \pi_N)$ in the $6N$-dimensional N-particle phase space Γ. The classical analogue of the v. Neumann entropy is the Gibbs entropy
$$S_{\text{Gibbs}} = -k \int \mathrm{d}^6\pi_1 \ldots \mathrm{d}^6\pi_N \, W(\pi_1 \ldots \pi_N) \ln W(\pi_1 \ldots \pi_N) \tag{2.5}$$
which, as expected, stays constant. Instead, one effectively describes the system by a probability density
$$w(\pi) = \int \mathrm{d}^6\pi_2 \ldots \mathrm{d}^6\pi_N \, W(\pi_1, \pi_2 \ldots \pi_N) \tag{2.6}$$

in the 6-dimensional single-particle phase space μ. It is the associated entropy

$$S_{\mathrm{rel}} = -kN \int \mathrm{d}^6\pi\, w(\pi) \ln w(\pi) \quad , \tag{2.7}$$

also known as the Boltzmann entropy, which increases! Hence the irreversibility paradox is avoided because the relevant entropy S_{rel} whose increase one observes is different from the Gibbs (alias v. Neumann) entropy.

The entropy of a probability distribution can be given an intuitive interpretation in the framework of information theory [52, 53, 56, 57, 58, 59, 60, 51]. It may be regarded as the amount of missing information as to the pure state of the system. Thus the v. Neumann entropy measures the amount of missing information as to the pure state of the system if one is given the Γ-space distribution W; in contrast, the relevant entropy measures the amount of missing information if one is only given the μ-space distribution w. Clearly, the description with w is less accurate, since it does not contain any information about particle correlations. Going from the full description with W to the less accurate description with w thus inevitably entails discarding information about the system; in this particular case, it is the information about particle correlations. The fact that information has been discarded is reflected by the inequality

$$S_{\mathrm{rel}} \geq S_{\mathrm{v.N.}} \quad . \tag{2.8}$$

These considerations carry over to arbitrary isolated systems. Generally, the v. Neumann entropy and the observed 'relevant' entropy refer to two different levels of description: the full description (classically by a distribution in Γ-space, quantum mechanically by the full statistical operator); and a reduced description (classically by, e.g., a distribution in a smaller phase space, quantum mechanically by a reduced statistical operator). These levels of description are referred to as microscopic and mesoscopic, respectively [61]. The transition from the microscopic to a mesoscopic level of description amounts to a coarse graining. Inevitably, information about certain degrees of freedom gets thereby lost. In the mesoscopic description, one retains information only about some selected observables which are deemed relevant.

In principle, there are many possible choices for the set of 'relevant observables,' each choice leading to a different level of description and consequently to a different relevant entropy. Three obvious choices come to mind: (i) *all* observables are deemed relevant, in which case the level of description becomes microscopic and the associated relevant entropy reduces to the v. Neumann entropy; (ii) only the constants of the motion are relevant, the associated relevant entropy thus coinciding with the equilibrium entropy of the system; (iii) those observables are considered relevant which are actually being measured in the experiment. None of these three categories, however, can accomodate the level of description used in the Boltzmann equation. There, *all* single-particle observables are taken to

relevant observables	level of description	relevant entropy S_{rel}
constants of the motion (C)	equilibrium	S_{eq}
experimentally measured observables (E)	experimental	S_{exp}
slowly varying observables (S)	Markovian	S_m
all observables (A)	microscopic	$S_{\mathrm{v.N.}}$

Table 2.1: Levels of description of a physical system and the associated relevant entropies.

be relevant, although in general only some of them are actually being measured (e.g., local energy and particle densities). The physical reason for that choice lies in the fact that single-particle observables vary much more slowly than particle correlations, the typical time scales being the time between two collisions vs. the duration of a collision; they thus constitute the set of slowly varying observables of the system. In fact, whenever a system exhibits a clear separation of time scales one defines a fourth (so-called 'Markovian') level of description which is comprised of all slowly varying observables. The four levels of description and the corresponding entropies are summarized in table 2.1.

Each level of description gives rise to a statistical prediction problem: given the present expectation values of the relevant observables, and possibly their past history, can one predict their future values? On the equilibrium level, this prediction problem is obviously trivial; and on the microscopic level, it can be solved — at least formally — by unitary evolution. On intermediate levels, however, predictions generally depend in a complicated manner on both the present values and the past history of the relevant observables. How far back into the past one has to reach in order to make a prediction is determined by the so-called *memory time.* Only if this memory time is small compared to the typical time scale on which the relevant observables evolve, can memory effects be neglected and predictions for the relevant observables be based solely on their present values. This approximation is known as the *Markovian limit;* which, in general, is justified only on the Markovian level of description (hence its name). Often one can expand around the Markovian limit, the resulting corrections giving rise to what are called *memory effects* [61, 15].

Clearly, a prediction cannot possibly contain more information than the data on which it is based. Thus in the Markovian limit, future expectation values of

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\rho_m(t_1) \arrow[r, "\text{(non-unitary) Markovian time evolution}"] \arrow[d, "\text{set}"'] & \rho_m(t_2) \\
\rho(t_1)=\rho_m(t_1) \arrow[r, "\text{unitary evolution}"] & \rho(t_2) \arrow[u, "\text{discard irrelevant information}"']
\end{tikzcd}$$

Figure 2.1: One 'proof' of the H-theorem. ρ_m *denotes the reduced state associated with the Markovian level of description;* ρ *denotes the full state. In the Markovian limit, the diagram must be commutative. Since the lower path involves discarding information, entropy must necessarily increase:* $S_m(t_2) \geq S_m(t_1)$, *Q.E.D.*

the relevant observables cannot carry more information than their present values (based on which they could be predicted). Applied iteratively, this means that the amount of information carried by the relevant observables can never increase; or, that the corresponding relevant entropy can never decrease — which is the *H-theorem.* An alternative argument for the H-theorem is sketched in figure 2.1 and discussed lucidly in Ref. [62] (with particular emphasis on the Boltzmann gas). It is important to note that the H-theorem, i.e., the increase of the relevant entropy, is a consequence of the Markovian limit and as such only applies to the Markovian level of description. This reveals the fundamental importance of the Markovian level.

While the constancy of the v. Neumann entropy shows that the full (microscopic) description keeps its information content, the growth of the relevant (Markovian) entropy indicates that the amount of information carried by the slowly varying observables steadily decreases. The only possible explanation for this phenomenon is that in the course of time, information about the system is being transferred from slow to fast degrees of freedom. This flow of information is perceived as irreversible. However, the information is not lost; it only becomes inaccessible to a certain level of description. As a consequence, the mesoscopic description loses predictive power. In the example of the dilute classical gas, information is being transferred from single-particle observables to many-particle correlations. Since in general these correlations will not be measured, part of the information about the system becomes experimentally inaccessible.

The various levels of description are not entirely independent. For example, the set of slowly varying observables contains by definition all constants of the motion. Typically it also contains all experimentally measured observables, because slow variation is usually necessary to make an observable measurable in practice. These and other relationships between different levels are illustrated in figure 2.2.

The typical behavior of the corresponding entropies is shown in figure 2.3. Both the v. Neumann and the equilibrium entropy remain constant in time. In contrast, the Markovian entropy increases monotonically (H-theorem); it is this monotonic increase of the Markovian entropy which reflects the *thermalization* of the system. No general statement can be made about the behavior of the experimental entropy, however; it may or may not have local extrema. That the experimental entropy may actually decrease (temporarily), can be illustrated in a simple example [61]. Imagine, for instance, two drops of a very dilute gas, each initially in local equilibrium, moving towards one another with large average velocities. Let us assume that only the local particle, energy, and momentum densities can be measured experimentally. On this reduced level of description, the associated experimental entropy will sharply increase as soon as the two drops begin to overlap; but after the encounter, it will *fall back* to practically its initial value. This decrease reflects the non-Markovian nature of the experimental level of description: the values of the local densities at time $t + \mathrm{d}t$ depend not only on their values at time t, but also on their past history before the encounter. In contrast, the Boltzmann entropy — which is associated with the Markovian level of description — will increase monotonically throughout the evolution of the system.

Although the Markovian entropy increases monotonically, it need not always approach the equilibrium entropy as $t \to \infty$. However, if it does, then the thermalization is *complete;* in which case only information about the constants of the motion is retained. Typically, the Markovian level of description contains the experimental level, so that $S_m \leq S_{\mathrm{exp}} \leq S_{\mathrm{eq}}$. Complete thermalization on the Markovian level then implies complete thermalization on the experimental level.

Understanding the thermalization properties of a physical system has now become entirely a matter of finding the Markovian level of description, and hence of identifying time scales. Any study of irreversible phenomena must therefore proceed by

- identifying all time scales,
- finding the Markovian level of description, and
- finding all conservation laws and thus all constants of the motion; i.e., finding the equilibrium level of description.

This knowledge can be used to obtain

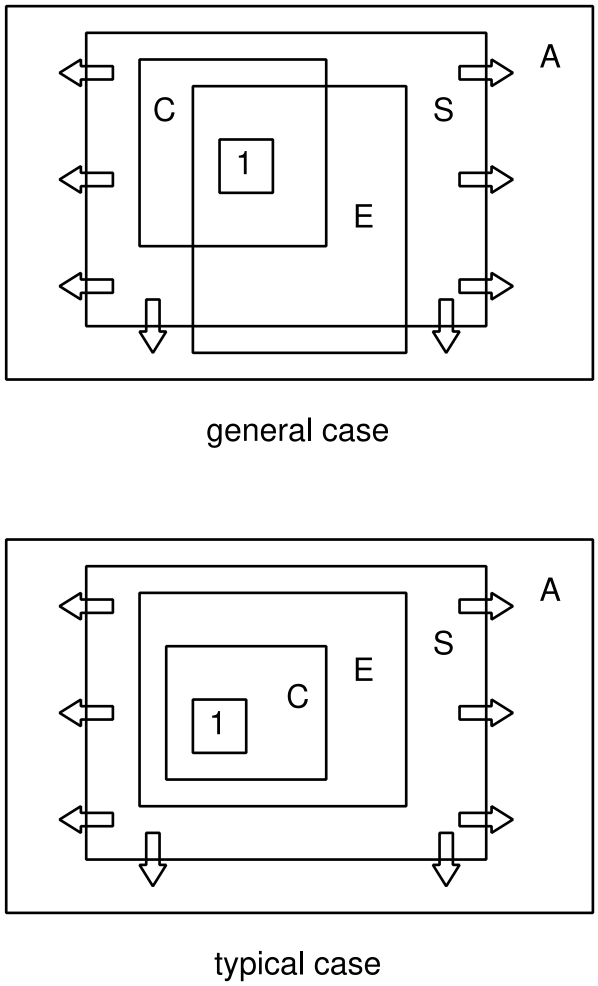

Figure 2.2: Levels of description of a physical system. 1 *denotes the unit operator; the arrows indicate the flow of information in the Markovian limit (H-theorem).*

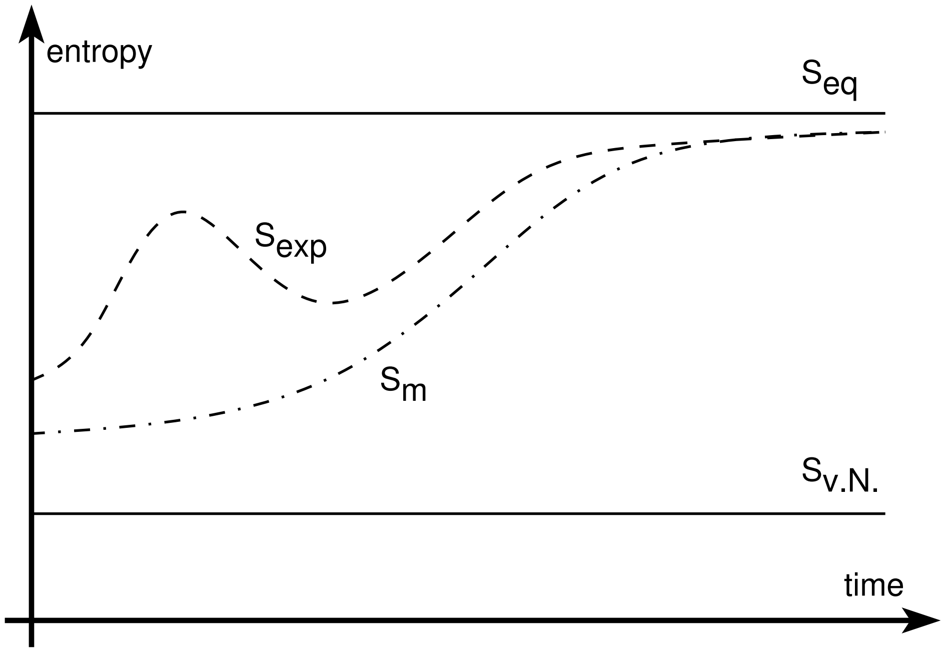

Figure 2.3: An example for the behavior of various entropies. Both the v. Neumann and the equilibrium entropy remain constant in time. The entropy S_m increases monotonically in the Markovian limit (H-theorem). No such general statement can be made about the behavior of the experimental entropy; it may or may not have local extrema. However, the experimental level of description is typically contained in the Markovian level, hence $S_{\rm exp} \geq S_m$.

- the rate of increase of the relevant (Markovian) entropy, and
- the equilibration time (provided the thermalization is complete).

The time scales, and hence the thermalization properties, of a system generally depend on

- features of the dynamics (i.e., the form of the interaction Lagrangian),
- coupling constants, and
- macroscopic parameters (e.g., density, pressure, etc.).

Chapter 3

The Projection Method

When dealing with a complicated quantum system such as a gas or liquid, one is typically neither interested in, nor capable of, describing the time evolution of all its microscopic properties. Instead, one is content with describing the evolution of only a few selected observables $\{G_a\}$, namely the relevant observables. To describe their time evolution it is not necessary to solve the Liouville-v. Neumann equation for the full statistical operator ρ; in fact, it would be foolish to attempt to do so. The reason is that the full statistical operator contains a lot of information which is not at all related to the relevant observables. This unnecessary 'baggage' complicates calculations without providing any additional insight into the behavior of the relevant observables. It is such irrelevant information which — for the sake of simplifying calculations — must be discarded. There are various ways of achieving that, one of the most powerful (and most general) being the projection method. This method was pioneered by Nakajima [63] and Zwanzig [64, 65, 66, 67, 68], and further developed by various other authors [69, 70, 71, 72, 73, 74, 75, 76]. A related method can be found in [77].

3.1 Liouville Space

The set of linear operators acting on Hilbert space is itself a vector space, called the Liouville space. Following the conventions of Petrosky and Prigogine [78], vectors in Liouville space are denoted by double-kets $|\cdot\rangle\rangle$. The relevant observables G_a now become vectors $|G_a\rangle\rangle$. Liouville space is endowed with an inner product

$$\langle\langle A|B\rangle\rangle := \mathrm{tr}\,(A^\dagger B) \quad . \tag{3.1}$$

An orthonormal basis in Liouville space can be constructed from an orthonormal basis $\{|n\rangle\}$ in Hilbert space; it is given by the so-called dyadic operators

$$|n;m\rangle\rangle := |\,|n\rangle\langle m|\,\rangle\rangle \quad . \tag{3.2}$$

Matrix elements of Hilbert space operators may then be written as

$$A_{nm} = \langle\langle n;m|A\rangle\rangle \quad . \tag{3.3}$$

The inner product allows one to write the expectation value of an arbitrary (possibly non-Hermitian) operator A as

$$\langle A \rangle = \langle\!\langle \rho | A \rangle\!\rangle \quad , \tag{3.4}$$

the state ρ thus being a linear functional ('double-bra') $\langle\!\langle \rho |$ on Liouville space. In particular, a pure state corresponding to a Hilbert space vector $|\psi\rangle$ is denoted by

$$\langle\!\langle \psi | := \langle\!\langle \, |\psi\rangle\langle\psi| \, | \quad . \tag{3.5}$$

Linear operators $\mathcal{O}$ acting on vectors in Liouville space are called 'super-operators.' One example for such a super-operator is the so-called Liouville operator (or Liouvillian) $\mathcal{L}(t)$ defined by

$$\begin{aligned} \langle\!\langle A | \mathcal{L}(t) | B \rangle\!\rangle &:= \hbar^{-1} \langle\!\langle A | [H(t), B] \rangle\!\rangle \\ &= \hbar^{-1} \langle\!\langle [H(t), A] | B \rangle\!\rangle \quad , \end{aligned} \tag{3.6}$$

$H(t)$ being the Hamiltonian of the system. With its help (and remembering that the inner product is antilinear in its first argument) the Liouville-v. Neumann equation (2.3) can be elegantly rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\!\langle \rho(t) | = i \langle\!\langle \rho(t) | \mathcal{L}(t) \quad . \tag{3.7}$$

This equation can be integrated formally by an evolution (super-)operator $\mathcal{U}(t_0, t)$,

$$\langle\!\langle \rho(t) | = \langle\!\langle \rho(t_0) | \mathcal{U}(t_0, t) \quad , \tag{3.8}$$

which satisfies

$$\frac{\partial}{\partial t} \mathcal{U}(t_0, t) = i \mathcal{U}(t_0, t) \mathcal{L}(t) \quad , \quad \mathcal{U}(t_0, t_0) = 1 \quad . \tag{3.9}$$

Hermitian conjugation of a super-operator with respect to the Liouville inner product is denoted by 'overbarring,' i.e.,

$$\langle\!\langle A | \mathcal{O} B \rangle\!\rangle = \langle\!\langle \overline{\mathcal{O}} A | B \rangle\!\rangle \quad . \tag{3.10}$$

Since the Liouville operator is by definition Hermitian, $\overline{\mathcal{L}}(t) = \mathcal{L}(t)$, the corresponding evolution operator $\mathcal{U}(t_0, t)$ is unitary: $\overline{\mathcal{U}}\mathcal{U} = \mathcal{U}\overline{\mathcal{U}} = 1$.

3.2 Projectors

The relevant observables $\{G_a\}$ span a (relatively small) subspace of the Liouville space. This subspace will be referred to as the *level of description*. The term 'level of description,' frequently used in chapter 2, thus acquires a precise mathematical meaning. To project the motion of the system onto this level of description is the basic idea behind the projection method. In the following, the relevant observables (and therefore the level of description) will be assumed to have no explicit time-dependence.

A projector in Liouville space is a super-operator $\mathcal{P}$ satisfying $\mathcal{P}^2 = \mathcal{P}$; it need not be Hermitian. If $\mathcal{P}$ is a projector then so is its complement $\mathcal{Q} := 1 - \mathcal{P}$. For our purposes we consider projectors which project arbitrary vectors in Liouville space onto the level of description; i.e.,

$$\mathcal{P}|A\rangle\!\rangle = |A\rangle\!\rangle \quad \text{iff} \quad |A\rangle\!\rangle \in \text{span}\{|G_a\rangle\!\rangle\} \quad . \tag{3.11}$$

As the projection need not be orthogonal, there are various possible choices for $\mathcal{P}$. In general, one allows for a dependence of the projector on the expectation values of the relevant observables,

$$g_a(t) := \langle G_a \rangle(t) = \langle\!\langle \rho(t)|G_a\rangle\!\rangle \quad , \tag{3.12}$$

thus making it an implicit function of time: $\mathcal{P} \equiv \mathcal{P}[g_a(t)]$ (cf. figure 3.1). Since the level of description itself remains fixed, a second projection will have no effect — even at a different time:

$$\mathcal{P}[g_a(t_2)]\mathcal{P}[g_a(t_1)] = \mathcal{P}[g_a(t_1)] \quad \forall\, t_1, t_2 \quad . \tag{3.13}$$

Alternatively, the projection may be viewed in the space of linear functionals on Liouville space (figure 3.2). In particular, the state $\rho(t)$ is projected onto what is called its *relevant part*

$$\langle\!\langle \text{rel}(t)| := \langle\!\langle \rho(t)|\mathcal{P}[g_a(t)] \quad . \tag{3.14}$$

Note that in general, this relevant part need not have all the properties (2.1) of a physical state; although one usually chooses the projector $\mathcal{P}$ such that it does. By definition, $\mathcal{P}|G_a\rangle\!\rangle = |G_a\rangle\!\rangle$ implies

$$\langle\!\langle \text{rel}(t)|G_a\rangle\!\rangle = g_a(t) \quad , \tag{3.15}$$

which shows that the relevant part carries all information about the expectation values of the relevant observables. The other component of the full state $\rho(t)$,

$$\langle\!\langle \text{irr}(t)| := \langle\!\langle \rho(t)|\mathcal{Q}[g_a(t)] \quad , \tag{3.16}$$

is referred to as its *irrelevant part.*

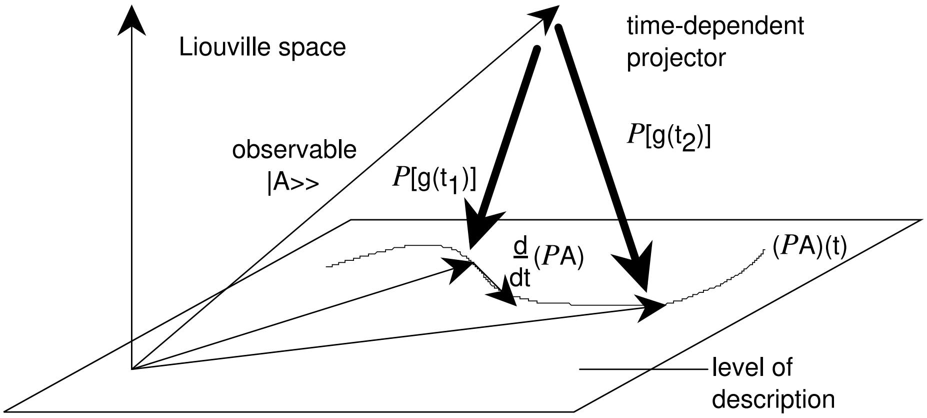

Figure 3.1: Arbitrary observables are projected onto the level of description. The projection need not be orthogonal and may vary in time.

For the further development of the projection method it will be necessary to impose some restriction on the possible time-dependence of the projectors. More precisely, one requires that

$$\langle\!\langle\rho(t)|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[g_a(t)] = \langle\!\langle\rho(t)|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}[g_a(t)] = 0 \quad . \tag{3.17}$$

This rather technical condition — which is purely a convention and has no physical significance — is intended to simplify subsequent calculations. It is in fact satisfied for most common choices for $\mathcal{P}$, including all the particular projectors which we will employ. For brevity, projectors will from now on be denoted by $\mathcal{P}(t)$ and $\mathcal{Q}(t)$.

While $\mathcal{U}(t_0, t)$ is the evolution operator corresponding to the Liouvillian $\mathcal{L}(t)$, let $\mathcal{T}(t_0, t)$ be the evolution operator corresponding to

$$\mathcal{L}_{\mathrm{irr}}(t) := \mathcal{Q}(t)\mathcal{L}(t)\mathcal{Q}(t) \quad , \tag{3.18}$$

i.e.,

$$\frac{\partial}{\partial t}\mathcal{T}(t_0, t) = i\,\mathcal{T}(t_0, t)\mathcal{L}_{\mathrm{irr}}(t) \quad , \quad \mathcal{T}(t_0, t_0) = 1 \quad . \tag{3.19}$$

This definition implies that also

$$\frac{\partial}{\partial t'}\mathcal{T}(t', t) = -i\,\mathcal{L}_{\mathrm{irr}}(t')\mathcal{T}(t', t) \quad . \tag{3.20}$$

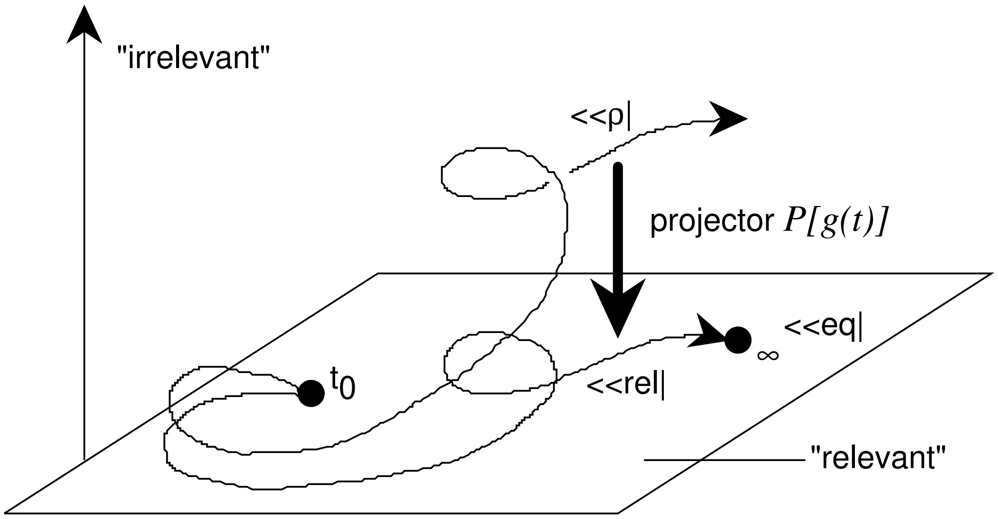

Figure 3.2: Schematic view of the projection in the space of linear functionals (states) on Liouville space. At every time t, the full statistical operator is projected onto its relevant part, the trajectory of states $\rho(t)$ thus being mapped onto a lower-dimensional subspace. In general, however, both the projector $\mathcal{P}$, and hence the subspace onto which it maps, may be time-dependent. If on the given level of description the system thermalizes completely, then the relevant part of the statistical operator eventually approaches an equilibrium distribution $\langle\!\langle \mathrm{eq}|$.

Since $\mathcal{Q}$ projects out the irrelevant component of an observable or state, the operators $\mathcal{L}_{\mathrm{irr}}$, and hence $\mathcal{T}$, may be pictured as describing the evolution of the *irrelevant* degrees of freedom of the system. (Note that in contrast to $\mathcal{U}$, the operator $\mathcal{T}$ need not be unitary.)

3.3 Equation of Motion for the Relevant Observables

Based on the Liouville-v. Neumann equation and on the definitions of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{T}$, I now derive an equation of motion for the expectation values of the relevant observables which is at the heart of the projection method. Starting from the identity

$$\begin{aligned} \langle\!\langle \mathrm{irr}(t)| &= \langle\!\langle \mathrm{irr}(t_0)|\mathcal{T}(t_0,t) + \\ &+ \int_{t_0}^{t} \mathrm{d}t' \left[\left(\frac{\mathrm{d}}{\mathrm{d}t'} \langle\!\langle \mathrm{irr}(t')| \right) \mathcal{T}(t',t) + \langle\!\langle \mathrm{irr}(t')| \frac{\partial}{\partial t'} \mathcal{T}(t',t) \right] \end{aligned} \tag{3.21}$$

and using (3.20) as well as the Liouville-v. Neumann equation and the technical condition (3.17), we obtain

$$\langle\!\langle \mathrm{irr}(t)| = \langle\!\langle \mathrm{irr}(t_0)|\mathcal{T}(t_0,t) + i \int_{t_0}^{t} \mathrm{d}t' \, \langle\!\langle \mathrm{rel}(t')|\mathcal{L}(t')\mathcal{Q}(t')\mathcal{T}(t',t) \quad . \tag{3.22}$$

We may multiply this equation with $\mathcal{Q}(t)$ from the right without changing the left-hand side. The equation of motion for the expectation values of the relevant observables can then be written as

$$\begin{aligned} \frac{\mathrm{d}g_a}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \langle\!\langle \rho(t)|G_a\rangle\!\rangle \\ &= i\langle\!\langle \mathrm{rel}(t)|\mathcal{L}(t)|G_a\rangle\!\rangle + i\langle\!\langle \mathrm{irr}(t)|\mathcal{L}(t)|G_a\rangle\!\rangle \\ &= i\langle\!\langle \mathrm{rel}(t)|\mathcal{L}(t)|G_a\rangle\!\rangle - \int_{t_0}^{t} \mathrm{d}t' \, \langle\!\langle \mathrm{rel}(t')|\mathcal{L}(t')\mathcal{Q}(t')\mathcal{T}(t',t)\mathcal{Q}(t)\mathcal{L}(t)|G_a\rangle\!\rangle + \\ &\quad + i\langle\!\langle \mathrm{irr}(t_0)|\mathcal{T}(t_0,t)\mathcal{Q}(t)\mathcal{L}(t)|G_a\rangle\!\rangle \quad . \end{aligned} \tag{3.23}$$

This equation is exact; no physical assumptions or approximations have been made.

The above equation of motion is the desired result: except possibly for the irrelevant part of the initial state, all irrelevant degrees of freedom have been eliminated. This elimination, of course, has its price. Comparing the above equation with the Liouville-v. Neumann equation for the true state, we notice that now there are two additional terms: (i) an integral ('memory') term, containing contributions from all times between the initial and the present time; and (ii) a 'residual force' term describing the effect of irrelevant components in the initial state. The physical meaning of these two terms can be easily discerned if read from left to right: (i) At time $t' < t$ the relevant part, due to interaction ($\mathcal{L}$), picks up an irrelevant component (projected out by $\mathcal{Q}$) which subsequently evolves in time ($\mathcal{T}$) and, due to a second interaction ($\mathcal{L}$), acquires relevancy again, thus contributing to the expectation value of the relevant observable G_a at the present time t. (ii) The irrelevant part of the initial state evolves in time ($\mathcal{T}$) and, due to interaction ($\mathcal{L}$), acquires relevancy at the present time t. Often the level of description and the projector are chosen such that the initial state does not have any irrelevant component, in which case the residual force term is absent.

Recalling that $\langle\langle \mathrm{rel}(t)| = \langle\langle \rho(t)|\mathcal{P}(t)$, and $\mathcal{P}(t)$ projects arbitrary observables onto the subspace spanned by $\{|G_a\rangle\rangle\}$, the above equation of motion has the structure

$$\dot{g_a}(t) = i\Omega_a^b(t)g_b(t) - \int_{t_0}^{t} \mathrm{d}t'\,\gamma_a^b(t,t')g_b(t') + f_a(t) \quad . \tag{3.24}$$

(Here I employed a summation convention.) Since the coefficients Ω, γ and f may themselves be non-trivial functions of the expectation values $\{g_c\}$, this is in general a system of nonlinear coupled integro-differential equations for $\{g_a(t)\}$. At first sight, such a system of equations may seem even more complicated and awkward than the original Liouville-v. Neumann equation; and indeed, in many cases it is just as hard to solve *exactly*. However, the above equation of motion serves as a much better starting point for various approximations — to be discussed shortly — which may greatly simplify the problem. In fact, many of the well-known equations of non-equilibrium statistical mechanics can be derived from there, either as special cases or with the help of suitable approximations. Two examples that I will discuss in detail are the Robertson equation and, later on, the quantum Boltzmann equation; other examples being the Nakajima-Zwanzig, Master, time-dependent Hartree-Fock, and Langevin equations [59].

3.4 Approximations

In the following I list some common approximations to equation (3.23). In order to streamline the notation I denote the kernel of the memory term by

$$\mathcal{K}(t',t) := \mathcal{P}(t')\mathcal{L}(t')\mathcal{Q}(t')\mathcal{T}(t',t)\mathcal{Q}(t)\mathcal{L}(t)\mathcal{P}(t) \quad . \tag{3.25}$$

- $\mathcal{K} = 0$ **Approximation.** The crudest approximation that can be made is to simply ignore the memory term, i.e., to set $\mathcal{K} = 0$. Then the equation of motion for the expectation values simply becomes

  $$\dot{g_a}^{(1)}(t) = i\langle\langle \mathrm{rel}(t)|\mathcal{L}(t)|G_a\rangle\rangle + \text{residual force} \quad . \tag{3.26}$$

  One prominent application is to a system of (nonrelativistic) fermions with effective 4-fermion interaction: Choosing the observation level to be the set of all single-particle observables and employing the Kawasaki-Gunton projection operator (see section 3.6), the $\mathcal{K} = 0$ approximation yields the time-dependent Hartree-Fock equations [59]. In that case the $\mathcal{K} = 0$ approximation is equivalent to a mean field approximation.

- **Perturbation Expansion.** Often the Liouville operator can be split into a free part and an interaction part,

$$\mathcal{L}(t) = \mathcal{L}^{(0)}(t) + \alpha\mathcal{V}(t) \tag{3.27}$$

(corresponding to $H = H^{(0)} + \alpha V$), α being the coupling constant. If free evolution does not mix relevant and irrelevant degrees of freedom, i.e., if

$$[\mathcal{L}^{(0)}(t), \mathcal{P}(t)] = 0 \quad , \tag{3.28}$$

then in the memory term, $\mathcal{PLQ} = \alpha \cdot \mathcal{PVQ}$ and $\mathcal{QLP} = \alpha \cdot \mathcal{QVP}$. The memory term is therefore at least of order $O(\alpha^2)$; in particular, it does not contribute in first order perturbation theory. (First order perturbation theory is equivalent to the $\mathcal{K} = 0$ approximation.) To obtain higher order contributions, one has to expand

$$\mathcal{Q}(t')\mathcal{T}(t', t)\mathcal{Q}(t) = \mathcal{Q}(t')\mathcal{T}^{(0)}(t', t)\mathcal{Q}(t) + O(\alpha) \tag{3.29}$$

where $\mathcal{T}^{(0)}$ is a time-ordered exponential of $\mathcal{L}^{(0)}_{\mathrm{irr}}(t)$. However, using the property (3.13) of projectors, all the $\mathcal{Q}$'s appearing in this time-ordered exponential can be shuffled to the right and absorbed into $\mathcal{Q}(t)$. Hence $\mathcal{T}^{(0)}$ may simply be replaced by $\mathcal{U}^{(0)}$, yielding, to second order,

$$\begin{aligned} \dot{g_a}^{(2)}(t) \;=\; & i\langle\!\langle \mathrm{rel}(t)|\mathcal{L}(t)|G_a\rangle\!\rangle - \\ & -\alpha^2 \int_{t_0}^{t} \mathrm{d}t'\, \langle\!\langle \mathrm{rel}(t')|\mathcal{V}(t')\mathcal{Q}(t')\mathcal{U}^{(0)}(t', t)\mathcal{Q}(t)\mathcal{V}(t)|G_a\rangle\!\rangle + \\ & + \text{residual force} \quad . \end{aligned} \tag{3.30}$$

- **Master Approximation.** If we assume that neither the projectors $\mathcal{P}$, $\mathcal{Q}$ nor the Liouville operator $\mathcal{L}$ are time-dependent, then the kernel $\mathcal{K}(t', t)$ becomes a function of the time difference only:

$$\mathcal{K}(t', t) = \mathcal{K}(t - t') \quad , \tag{3.31}$$

with

$$\mathcal{K}(t) = \mathcal{PLQ}\exp(i\mathcal{L}_{\mathrm{irr}}t)\mathcal{QLP} \quad . \tag{3.32}$$

Equation (3.23) then reduces to the Generalised Master equation. Two well-known applications of this approximation are: (i) to a system in a heat bath b whose temperature is assumed to be constant — in which case the projector $\mathcal{P}A = 1_b \otimes \mathrm{tr}_b[Z^{-1}\exp(-\beta H^b)A]$ is time-independent; and (ii) to any system for which we choose the observation level to consist of the free Hamiltonian $H^{(0)}$ and of every observable that commutes with it, the projection $\langle\langle\rho| \to \langle\langle\rho|\mathcal{P}$ thus removing all coherence of the state with respect to $H^{(0)}$. Then $\mathcal{P}$ is by definition not time-dependent, and the resulting equation is known as the Pauli Master equation.

- **Markovian Limit.** As I discussed in chapter 2, there are generally two kinds of time scales: (i) the time scale τ_{rel} (or several time scales $\tau_{\mathrm{rel}}^{(i)}$) on which the expectation values of the relevant observables evolve; and (ii) the memory time τ_{mem}, characterising the length of the time interval which contributes significantly to the memory term. Often the memory kernel $\mathcal{K}(t',t)$ is proportional to some distribution (e.g., Gaussian, Lorentzian) in $(t-t')$, in which case the memory time can simply be identified with the width of that distribution. Symbolically:

$$\mathcal{K}(t',t) \approx 0 \quad \text{iff} \quad (t-t') > \tau_{\mathrm{mem}} \quad . \tag{3.33}$$

In case $\tau_{\mathrm{mem}} \ll \tau_{\mathrm{rel}}$ one may assume that in the memory term, $\langle\langle \mathrm{rel}(t')| \approx \langle\langle \mathrm{rel}(t)|$ and therefore (ignoring the residual force)

$$\dot{g_a}(t) \approx \langle\langle \mathrm{rel}(t)| \left[i\mathcal{L}(t) - \int_{t_0}^{t} \mathrm{d}t'\, \mathcal{K}(t',t) \right] |G_a\rangle\rangle \quad . \tag{3.34}$$

The change of the expectation values of the relevant observables now only depends on their present values, their past history being completely ignored. This is the Markovian limit. As we will see, a Markovian limit is involved in the derivation of all Boltzmann-type transport equations. Note that nevertheless the kernel $\mathcal{K}$ still contributes — ignoring the state's past history does not mean that the memory term is eliminated altogether. One may expand around the Markovian limit:

$$\langle\langle \mathrm{rel}(t')| = \langle\langle \mathrm{rel}(t)| - (t-t')\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \mathrm{rel}(t)| + O(\tau_{\mathrm{mem}}^2) \quad . \tag{3.35}$$

With the definitions

$$\mathcal{K}_{(0)}(t) \;:=\; \int_{t_0}^{t} \mathrm{d}t'\, \mathcal{K}(t',t) \quad , \tag{3.36}$$

$$\mathcal{K}_{(1)}(t) \;:=\; \int_{t_0}^{t} \mathrm{d}t'\, (t-t')\mathcal{K}(t',t) \tag{3.37}$$

— where $\mathcal{K}_{(1)}/\mathcal{K}_{(0)}$ is $O(\tau_{\mathrm{mem}})$ — one then finds

$$\begin{aligned} \dot{g_a}(t) \;&=\; \langle\langle \mathrm{rel}(t)| \Big[i\mathcal{L}(t) - \mathcal{K}_{(0)}(t) + i\mathcal{L}(t)\mathcal{K}_{(1)}(t) - \\ &\quad -\mathcal{K}_{(0)}(t)\mathcal{K}_{(1)}(t) \Big] |G_a\rangle\rangle + O(\tau_{\mathrm{mem}}^2) \quad . \end{aligned} \tag{3.38}$$

In a perturbation expansion the kernel $\mathcal{K}$ is of second order in the coupling constant α. Thus any non-Markovian correction is at least of order $O(\alpha^2 \tau_{\text{mem}})$. In case $\mathcal{P}(t)\mathcal{L}(t)\mathcal{P}(t) = 0$, the term $\mathcal{L}\mathcal{K}_{(1)}$ does not contribute, and hence the correction becomes $O(\alpha^4 \tau_{\text{mem}})$.

- **Quasistationary Approximation.** For times t with $(t - t_0) \gg \tau_{\text{mem}}$ the exact 'location' of the initial time t_0 becomes irrelevant. One may then let $t_0 \to -\infty$, yielding the memory term

$$-\int_0^\infty \mathrm{d}\tau \, \langle\!\langle \mathrm{rel}(t-\tau) | \mathcal{K}(t-\tau, t) | G_a \rangle\!\rangle \quad . \tag{3.39}$$

This removal of the 'cut-off' t_0 restores temporal homogeneity and hence energy conservation.*

In many practical applications of the projection method several of the above approximations have to be combined in order to make a solution feasible.

3.5 Example: Resonance Decay

In this section, I consider the decay of a single resonance $|i\rangle$ weakly coupled to a continuum of states $\{|\omega\rangle\}$ (figure 3.3). The Hamiltonian is of the form

$$H = H^{(0)} + \alpha V \quad , \tag{3.40}$$

the perturbation having non-vanishing matrix elements

$$V_{i\omega} = \langle i | V | \omega \rangle = \langle \omega | V | i \rangle^* \quad . \tag{3.41}$$

For simplicity, all other matrix elements of V are taken to be zero. I wish to describe the time evolution of the occupation probability

$$p(t) := |\langle i | \psi(t) \rangle|^2 \quad , \tag{3.42}$$

given that the system is initially in the resonance state:

$$|\psi(t_0)\rangle = |i\rangle \quad . \tag{3.43}$$

Despite its simplicity, this example already exhibits all the essential features of a non-equilibrium process. Of course, the decay of a resonance can be described by conventional methods [79]; here, however, I want to use it as a testing ground for the projection method (see, e.g., [80]). I will lay out an agenda that can also be applied to more complicated problems.

*unless, of course, there is an explicit time-dependence of the Hamiltonian.

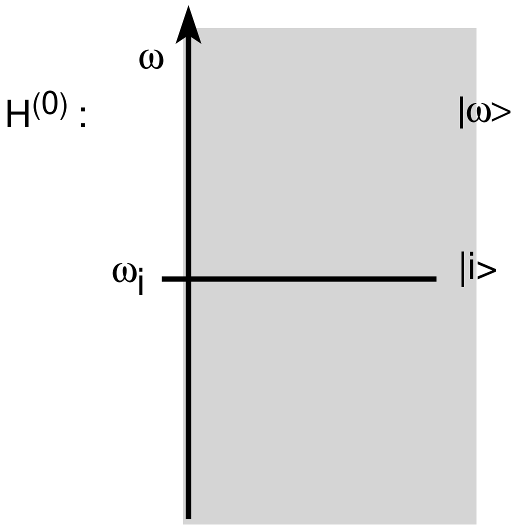

Figure 3.3: Resonance decay: spectrum of the unperturbed Hamiltonian.

- **Level of description.** The only relevant observable is $|i\rangle\langle i|$; hence $\{|G_a\rangle\rangle\} = \{|i\rangle\rangle\}$. Its expectation value is $p(t) = \langle\langle\rho(t)|i\rangle\rangle$.

- **Projector.** One simple choice is $\mathcal{P} = |i\rangle\rangle\langle\langle i|$. It is obvious that this projector satisfies both $\mathcal{P}|i\rangle\rangle = |i\rangle\rangle$ and the condition (3.17).

- **Residual force.** By assumption, $\langle\langle\rho(t_0)| = \langle\langle i|$ and thus $\langle\langle\mathrm{irr}(t_0)| = 0$. Ergo the residual force term vanishes.

- **1st term.** The first term in the equation of motion (3.23) vanishes as well, because $\langle\langle i|\mathcal{L}|i\rangle\rangle \propto \mathrm{tr}\,(|i\rangle\langle i|[H, |i\rangle\langle i|]) = 0$.

- **Perturbation expansion.** Only the memory term remains. It will be calculated to lowest (i.e., second) order in the coupling constant α. The Liouville operator has the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \alpha\mathcal{V} \quad , \tag{3.44}$$

where $\mathcal{L}^{(0)}\mathcal{P} = \mathcal{P}\mathcal{L}^{(0)} = 0$ and thus trivially

$$[\mathcal{L}^{(0)}, \mathcal{P}] = 0 \quad . \tag{3.45}$$

Therefore we can take over the general result (3.30). It can be simplified further: (i) since $\mathcal{PVP} = 0$ all the $\mathcal{Q}$'s can be omitted; (ii) as $|i\rangle\rangle$ is invariant under free evolution, $\mathcal{U}^{(0)}\mathcal{V}$ may be replaced by the interaction-picture operator

$$\mathcal{V}^I(t-t') := \mathcal{U}^{(0)}(t',t)\mathcal{V}\mathcal{U}^{(0)}(t,t') \quad . \tag{3.46}$$

Substituting $\tau = t - t'$ finally yields

$$\dot{p}(t) = -\alpha^2 \int_0^{t-t_0} \mathrm{d}\tau \, \langle\langle \mathrm{rel}(t-\tau)|\mathcal{V}\mathcal{V}^I(\tau)|i\rangle\rangle \quad . \tag{3.47}$$

With $\mathcal{P} = |i\rangle\rangle\langle\langle i|$ and $\langle\langle \mathrm{rel}(t-\tau)|i\rangle\rangle = p(t-\tau)$, this implies

$$\dot{p}(t) = -\alpha^2 \int_0^{t-t_0} \mathrm{d}\tau \, p(t-\tau)\langle\langle i|\mathcal{V}\mathcal{V}^I(\tau)|i\rangle\rangle \quad . \tag{3.48}$$

The last factor in the integrand can be determined using (i) the definition $\mathcal{V}|i\rangle\rangle := \hbar^{-1}|[V,|i\rangle\langle i|]\rangle\rangle$ (analogous for $\mathcal{V}^I$); (ii) the completeness property

$$|i\rangle\langle i| + \int \frac{\mathrm{d}\omega}{2\pi}\,|\omega\rangle\langle\omega| = 1 \quad ; \tag{3.49}$$

and (iii) $V_{ii} = 0$. One finds

$$\langle\langle i|\mathcal{V}\mathcal{V}^I(\tau)|i\rangle\rangle = \frac{2}{\hbar^2}\mathrm{Re}\int \frac{\mathrm{d}\omega}{2\pi}\,|V_{i\omega}|^2 \exp[i(\omega-\omega_i)\tau] \quad . \tag{3.50}$$

- **Coupling width and memory time.** In general, the resonance couples to an entire 'band' of continuum states. This band has some width Δ, called the *coupling width* (see figure 3.4). For the sake of simplicity, let us parametrize the coupling as a Gaussian distribution

$$|V_{i\omega}|^2 = \frac{v^2}{\Delta}\exp\left[-\frac{(\omega-\omega_i)^2}{2\Delta^2}\right] \quad , \tag{3.51}$$

normalized such that $\mathrm{tr}\,(V^2) = \sqrt{2/\pi}v^2$. Then the frequency integration may be performed, yielding another Gaussian distribution for τ with width Δ^{-1}:

$$\langle\langle i|\mathcal{V}\mathcal{V}^I(\tau)|i\rangle\rangle = \frac{v^2}{\hbar^2}\sqrt{\frac{2}{\pi}}\exp\left[-\frac{\Delta^2\tau^2}{2}\right] \quad . \tag{3.52}$$

From this we obtain the memory kernel

$$\begin{aligned} \mathcal{K}(\tau) &= |i\rangle\rangle\langle\langle i|\mathcal{V}\mathcal{V}^I(\tau)|i\rangle\rangle\langle\langle i| \\ &= \frac{v^2}{\hbar^2}\sqrt{\frac{2}{\pi}}\exp\left[-\frac{\Delta^2\tau^2}{2}\right]\cdot|i\rangle\rangle\langle\langle i| \quad . \end{aligned} \tag{3.53}$$

Evidently, $\Delta^{-1} = \tau_{\mathrm{mem}}$; the memory time is given by the inverse coupling width.

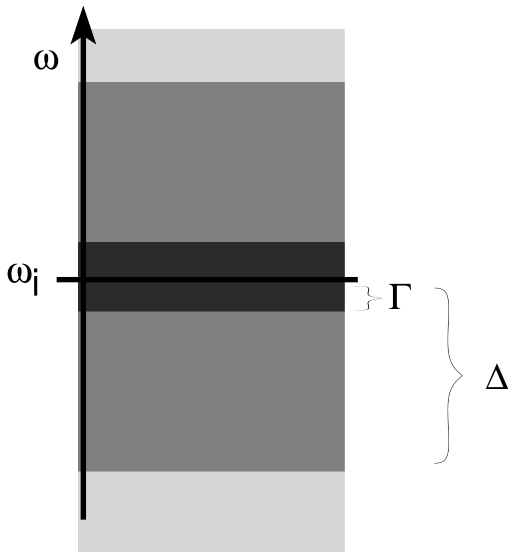

Figure 3.4: Resonance decay: decay width Γ *and coupling width* Δ*. The Markovian limit is justified in the narrow resonance approximation* ($\Gamma \ll \Delta$).

- **Markovian limit.** Provided the memory time is very small, one can (i) take the Markovian limit, $p(t-\tau) \to p(t)$, and (ii) do the quasistationary approximation, $(t-t_0) \to \infty$. Performing the τ-integration in (3.48) then yields
$$\dot{p}(t) = -\left(\frac{\alpha^2 v^2}{\hbar^2 \Delta}\right) p(t) \quad , \tag{3.54}$$
the usual exponential decay law. The coefficient $\Gamma := \alpha^2 v^2/\hbar^2\Delta$ is called the *decay width;* it determines the timescale $\Gamma^{-1} = \tau_{\rm rel}$ on which the expectation value $p(t)$ evolves. Clearly, the Markovian limit is justified only if $\Gamma^{-1} \gg \Delta^{-1}$ ('narrow resonance approximation'; cf. figure 3.4).

This simple example teaches some important lessons:

1. The memory time $\tau_{\rm mem}$ is related to the coupling width: $\tau_{\rm mem} = \Delta^{-1}$. Hence the memory time is determined by the *form* of the interaction only; it does not depend on its strength.

2. The relaxation time $\tau_{\rm rel}$, on the other hand, does depend on the strength of the interaction; it is of order $O(\alpha^{-2})$ in the coupling constant.

3. Therefore the Markovian limit becomes exact as $\alpha \to 0$ (while keeping Δ, v and hence $\alpha^2\tau_{\rm rel}$ fixed — the so-called Van Hove limit [81, 82, 83]).

3.6 Kawasaki-Gunton Projector, Relevant Entropy, and the Robertson Equation

On a given level of description there are, in general, many possible choices for the projector $\mathcal{P}[g_a(t)]$. They are restricted only by the conditions $\mathcal{P}|G_a\rangle\!\rangle = |G_a\rangle\!\rangle$ and (3.17). However, some choices may be more convenient than others. In this section, I will introduce one special choice that is widely applicable and particularly useful — the Kawasaki-Gunton projector [84].

The basic idea is that one wants the relevant part of the state to have the canonical form
$$\langle\!\langle \mathrm{rel}[g_a(t)]| = \langle\!\langle Z^{-1} \exp(-\lambda^a(t) G_a)| \tag{3.55}$$
at all times t. The relevant part is thus at all times determined using the Maximum Entropy Principle,
$$\langle\!\langle \rho(t)| \stackrel{g_a(t)}{\longrightarrow} \langle\!\langle \mathrm{rel}(t)| \quad , \tag{3.56}$$
given the (generally time-dependent) expectation values $g_a(t) \equiv \langle G_a\rangle(t)$ as constraints (cf. appendix B). The Lagrange parameters are adjusted so as to yield the correct expectation values for the relevant observables. (Naturally the Lagrange parameters, too, are functions of time.) With this definition, the relevant

part contains information pertaining to the relevant observables only; irrelevant information has been discarded entirely.

A projector which yields a canonical relevant part does indeed exist: it is the *Kawasaki-Gunton projector*

$$\mathcal{P}[g_a(t)] = |1\rangle\!\rangle\langle\!\langle \mathrm{rel}[g_a(t)]| - g_b \frac{\partial}{\partial g_b}|1\rangle\!\rangle\langle\!\langle \mathrm{rel}[g_a(t)]| + \frac{\partial}{\partial g_b}|G_b\rangle\!\rangle\langle\!\langle \mathrm{rel}[g_a(t)]| \qquad (3.57)$$

where $|1\rangle\!\rangle$ denotes the unit operator, $\langle\!\langle \mathrm{rel}|$ is the canonical state defined above, g_b stands for $g_b(t)$, the derivative $\partial/\partial g_b$ acts on $\langle\!\langle \mathrm{rel}|$, and a sum over b is implied. While it is obvious that $\mathcal{P}|G_a\rangle\!\rangle = |G_a\rangle\!\rangle$, some further thought is required to prove that the Kawasaki-Gunton projector also satisfies the condition (3.17) [59].

Employing the Kawasaki-Gunton projector has four potential advantages:

- Often the only information that is available about the initial state are the initial expectation values $\{\langle G_a\rangle(t_0)\}$. Then the initial state is itself determined via the Maximum Entropy Principle, and hence has just the canonical form (3.55). In that case the initial state has no irrelevant component, which in turn implies that there is no residual force. Eliminating the residual force term in this manner greatly simplifies the calculation.

- If all relevant observables commute, i.e., if $[G_a, G_b] = 0$, then the first term of the equation of motion vanishes as well,

$$\begin{aligned} \langle\!\langle \mathrm{rel}|\mathcal{L}|G_a\rangle\!\rangle &\propto \mathrm{tr}\,\Big(\exp(-\lambda^b G_b)[H, G_a]\Big) \\ &\propto \mathrm{tr}\,\Big([G_a, \exp(-\lambda^b G_b)]H\Big) \\ &= 0 \quad , \end{aligned} \qquad (3.58)$$

  independently of the Liouvillian. However, this conclusion must be taken with a grain of salt: its derivation depended on the cyclic permutability of operators under the trace, a property which in general only holds if both G_a and H are bounded [85]. (As a counterexample, consider $\mathrm{tr}\,(\hat{x}\hat{p}) - \mathrm{tr}\,(\hat{p}\hat{x}) = \mathrm{tr}\,[\hat{x}, \hat{p}] = \mathrm{tr}\,(i) \neq 0$.) Indeed, we will soon encounter an application — the quantum Boltzmann equation in the presence of an electric field — where the above conclusion does not apply.

- In the next chapter, I will study systems of interacting particles and use the projection method to derive an equation of motion for selected single-particle observables. In order to evaluate the memory term I will have to calculate expectation values $\langle\!\langle \mathrm{rel}|a_i^\dagger a_j a_k^\dagger a_l \ldots\rangle\!\rangle$ of products of several field operators. Only if the relevant part $\langle\!\langle \mathrm{rel}|$ has the so-called Hartree form (cf. appendix C), can Wick's theorem be applied and these expectation values be evaluated. The Hartree form is guaranteed if one employs the Kawasaki-Gunton projector.

Physical system	Level of description $\{G_a\}$	Relevant part $\rho_{\mathrm{rel}}(t)$	Discarded information
$\Sigma^1 \times \Sigma^2 \times \ldots$ several subsystems	all Σ^i-observables $(i = 1, 2, \ldots)$	$\rho^1(t) \otimes \rho^2(t) \otimes \ldots$	correlations between subsystems
many-particle system	all or selected single-particle observables	$Z^{-1} \exp\left(-\sum \lambda^{ij}(t) a_i^\dagger a_j\right)$	particle correlations
$S \times B$ system in heat bath	all S-observables and bath energy	$\rho^s(t) \otimes$ $\otimes Z^{-1} \exp\left(-\beta(t) H^b\right)$	internal degrees of freedom of the bath; system-bath correlations
(any system)	observable A (eigenvalues a_i, projectors P_i) and all observables commuting with A	$\sum_i P_i \rho(t) P_i$	coherence with respect to A

Table 3.1: Applications of the Kawasaki-Gunton projector.

- From a conceptual point of view the canonical form (3.55) is the most 'natural' choice for the relevant part, simply because — due to the entropy maximization involved — it contains no irrelevant information at all. For this very reason it is *this* choice for the relevant part which is used to define the *relevant entropy:*

  $$S_{\mathrm{rel}}[g_a(t)] := -k \,\mathrm{tr}\, \left(\rho_{\mathrm{rel}}[g_a(t)] \ln \rho_{\mathrm{rel}}[g_a(t)]\right) \quad . \tag{3.59}$$

  The relevant entropy is also known as the *accompanying entropy with respect to the level of description* (spanned by $\{G_a\}$). It measures, at all times t, the amount of missing information as to the pure state of the system if one is given the current expectation values $\{g_a(t)\}$ of the relevant observables.

Whenever the Kawasaki-Gunton projector is used then the equation of motion (3.23) goes by the name of *Robertson equation* [71, 72, 73, 74, 75, 59]. Several applications are listed in table 3.1.

Chapter 4

The Quantum Boltzmann Equation

In this chapter, I employ the projection method to study in detail the evolution of a quantum system of interacting identical particles. With few modifications, I execute the agenda already laid out during my discussion of the resonance decay. These investigations lead to a derivation and careful analysis of the quantum Boltzmann equation. A detailed understanding of the quantum Boltzmann equation is necessary before any attempt (in chapter 5) to incorporate particle creation.

4.1 Preliminaries

I consider a quantum system of identical particles (either bosons or fermions) with mass m, spin s and electric charge q. (For electrons, $q = -e$.) Particles interact through binary ($2 \to 2$) collisions. In addition, they are accelerated by an external electric field $\vec{E}$. This external field is assumed to be time-independent and homogeneous. Here, its effect is restricted to acceleration; pair creation will be incorporated later (in chapter 5). Any modifications due to backreaction will be neglected.

Let us first choose a basis of single-particle states,

$$\{|i\rangle\} := \{|\vec{p}_i, m_i\rangle\} \quad , \tag{4.1}$$

labelled by the momentum $\vec{p}_i$ and a spin component m_i ($m_i = -s, \ldots, +s$). Note that since I will use the time-independent gauge for the electromagnetic field ($\vec{A} = 0$), there is no distinction between the canonical and the kinetic momentum. For the sake of simplicity, I assume the system to be confined to a large, but finite, volume Ω so that one can use the box normalization

$$\langle i|j\rangle = \delta_{ij} \tag{4.2}$$

and completeness property

$$\sum_i |i\rangle\langle i| = 1 \quad . \tag{4.3}$$

Here

$$\delta_{ij} := \delta_{\vec{p}_i\vec{p}_j}\delta_{m_i m_j} \quad \text{and} \quad \sum_i := \sum_{\vec{p}_i}\sum_{m_i} \quad . \tag{4.4}$$

For each basis state $|i\rangle$ one defines annihilation and creation operators a_i, $a_i^\dagger$, where a_i is a shorthand notation for $a(\vec{p}_i, m_i)$. These field operators satisfy commutation or anticommutation relations:

$$[a_i, a_j]_\mp = [a_i^\dagger, a_j^\dagger]_\mp = 0 \quad , \tag{4.5}$$

$$[a_i, a_j^\dagger]_\mp = \delta_{ij} \quad , \tag{4.6}$$

depending on whether they describe bosons (upper sign) or fermions (lower sign). The operator

$$N(\vec{p}_i) := \sum_{m_i} a_i^\dagger a_i \tag{4.7}$$

then corresponds to the number of particles with momentum $\vec{p}_i$; for both bosons and fermions, it satisfies the commutation relations

$$\begin{aligned} [a_k, N(\vec{p})] &= \delta_{\vec{p}_k\vec{p}}\, a_k \\ [a_k^\dagger, N(\vec{p})] &= -\delta_{\vec{p}_k\vec{p}}\, a_k^\dagger \quad . \end{aligned} \tag{4.8}$$

The Hamilton operator of the system is the sum of three terms: (i) the kinetic energy $H^{(0)}$; (ii) the potential energy in the external field, $q\Phi$; and (iii) a term αV which describes binary scattering (α being the respective coupling constant). Thus

$$\begin{aligned} H &= \underbrace{H^{(0)} + q\Phi} + \alpha V \\ &= H^{(E)} + \alpha V \quad . \end{aligned} \tag{4.9}$$

The kinetic energy and the potential energy in the external field together constitute the single-particle contribution $H^{(E)}$ ($\sim a^\dagger a$) to the Hamiltonian; whereas αV ($\sim a^\dagger a^\dagger a a$) is the two-particle contribution. In perturbation theory, it will be preferable to use this latter decomposition of H into the single-particle contribution and the two-particle contribution, i.e., $H = H^{(E)} + \alpha V$; rather than the decomposition into $H^{(0)}$ plus 'the rest.'

Under the influence of the homogeneous electric field $\vec{E}$ (but in the absence of collisions), the single-particle basis states evolve according to

$$U^{(E)}(\tau)|\vec{p}, m\rangle = \exp\left[-i\int_0^\tau \mathrm{d}\tau'\, \omega(\vec{p} + q\vec{E}\tau')\right] |\vec{p} + q\vec{E}\tau, m\rangle \quad . \tag{4.10}$$

Here $\omega := \epsilon/\hbar$ is the frequency associated with the kinetic energy $\epsilon(\vec{k})$, which may be either nonrelativistic or relativistic. Using $U^{(E)}$ rather than $U^{(0)}$, we may define *interaction picture* operators; in particular,

$$\begin{aligned} a^{\dagger I}(\vec{p},m;\tau) &= \exp\left[+i\int_0^\tau \mathrm{d}\tau'\,\omega(\vec{p}-q\vec{E}\tau')\right] a^\dagger(\vec{p}-q\vec{E}\tau,m) \\ a^{I}(\vec{p},m;\tau) &= \exp\left[-i\int_0^\tau \mathrm{d}\tau'\,\omega(\vec{p}-q\vec{E}\tau')\right] a(\vec{p}-q\vec{E}\tau,m) \quad . \end{aligned} \tag{4.11}$$

As an immediate consequence,

$$N^I(\vec{p};\tau) = N(\vec{p}-q\vec{E}\tau) \quad . \tag{4.12}$$

The two-particle interaction has the form

$$V = \frac{1}{4}\sum_{ijkl}\langle lk|V|ji\rangle_\pm a_l^\dagger a_k^\dagger a_j a_i \quad , \tag{4.13}$$

where $\langle\ldots\rangle_\pm$ denotes the (anti-)symmetrized matrix element

$$\langle lk|V|ji\rangle_\pm := \langle lk|V|ji\rangle \pm \langle kl|V|ji\rangle \quad . \tag{4.14}$$

Defining

$$V_{lk|ji} := \langle lk|V|ji\rangle_\pm a_l^\dagger a_k^\dagger a_j a_i \tag{4.15}$$

with the symmetries

$$V_{lk|ji} = V_{kl|ji} = V_{lk|ij} \quad , \quad V_{lk|ji}^\dagger = V_{ij|kl} \quad , \tag{4.16}$$

one can also write

$$V = \frac{1}{4}\sum_{ijkl} V_{lk|ji} \quad . \tag{4.17}$$

In the limit of an infinite volume ($\Omega \to \infty$), the 2-body potential in ordinary space depends only on the particles' relative position; the dynamics is invariant under spatial displacements. Such spatial homogeneity implies momentum conservation:

$$V_{lk|ji} \neq 0 \quad \text{only if} \quad \vec{p}_i + \vec{p}_j = \vec{p}_k + \vec{p}_l \quad . \tag{4.18}$$

For future reference, let me calculate (using (4.8))

$$[V_{lk|ji}, N(\vec{p})] = (\delta_{\vec{p}_i\vec{p}} + \delta_{\vec{p}_j\vec{p}} - \delta_{\vec{p}_k\vec{p}} - \delta_{\vec{p}_l\vec{p}})V_{lk|ji} \quad . \tag{4.19}$$

This commutator vanishes whenever $V_{lk|ji}$ corresponds to zero momentum transfer. For example, if $\vec{p}_l = \vec{p}_j$ and hence $\vec{p}_k = \vec{p}_i$ (due to momentum conservation), then the Kronecker symbols add up to zero. More generally,

$$[V_{lk|ji}, N(\vec{p})] = 0 \quad \text{if} \quad l=j,\ l=i,\ k=j,\ \text{or}\ k=i \quad . \tag{4.20}$$

4.2 Applying the Projection Method

- I wish to describe the evolution of the momentum distribution, i.e., of the occupation numbers
$$n(\vec{p},t) := \langle N(\vec{p})\rangle(t) \quad . \tag{4.21}$$
The number operators $\{N(\vec{p})\}$ thus constitute the set of relevant observables, spanning the **level of description**.

- As the associated **projector** I choose the Kawasaki-Gunton projector, for the reasons discussed in the previous chapter. Then at all times t, the relevant part of the state has the canonical form
$$\langle\!\langle \mathrm{rel}(t)| = \langle\!\langle Z^{-1}\exp\left(-\sum_{\vec{p}}\lambda(\vec{p},t)N(\vec{p})\right)| \quad . \tag{4.22}$$

- Initially, at time t_0, the state of the system is characterized only by the expectation values $\{n(\vec{p},t_0)\}$; the initial state thus has the canonical form
$$\langle\!\langle \rho(t_0)| = \langle\!\langle Z^{-1}\exp\left(-\sum_{\vec{p}}\lambda_0(\vec{p})N(\vec{p})\right)| \quad . \tag{4.23}$$
Clearly, it has no irrelevant component. Therefore the **residual force** term vanishes.

In the (Robertson) equation of motion (3.23) we are left with the first term and the memory term. In the following subsections I will show that they correspond to the acceleration term and the collision term, respectively.

4.2.1 First Term

Since all relevant observables commute, $[N(\vec{p}),N(\vec{p}\,')] = 0$, one might immediately conclude that the first term must vanish (cf. section 3.6). However, we already cautioned that this conclusion may not always be trusted, so we better check explicitly.

There may be two contributions, from $\mathcal{L}^{(E)}$ and from $\alpha\mathcal{V}$. Let us first consider
$$i\langle\!\langle \mathrm{rel}|\alpha\mathcal{V}|N(\vec{p})\rangle\!\rangle \propto \sum_{ijkl}\langle\!\langle \mathrm{rel}|[V_{lk|ji},N(\vec{p})]\rangle\!\rangle \quad . \tag{4.24}$$

Due to the commutation relation (4.19), this reduces to a linear combination of expectation values $\langle V_{lk|ji}\rangle_{\mathrm{rel}}$. According to (4.20), only those terms contribute for which $\{l,k\} \neq \{j,i\}$; yet according to Wick's theorem (cf. appendix C), the expectation values can be non-zero only if $\{l,k\} = \{j,i\}$. Consequently the two-particle scattering $\alpha\mathcal{V}$ does not contribute at all to the first term of the Robertson equation.

There remains

$$\begin{aligned}
\dot{n}^{\mathrm{first}}(\vec{p},t) &= i\langle\!\langle \mathrm{rel}(t)|\mathcal{L}^{(E)}|N(\vec{p})\rangle\!\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \langle\!\langle \mathrm{rel}(t)|N^I(\vec{p};\tau)\rangle\!\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \langle\!\langle \mathrm{rel}(t)|N(\vec{p}-q\vec{E}\tau)\rangle\!\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} n(\vec{p}-q\vec{E}\tau,t) \\
&= -q\vec{E}\cdot\nabla_{\vec{p}}\, n(\vec{p},t) \\
&\equiv \dot{n}^{\mathrm{acc}}(\vec{p},t)
\end{aligned} \tag{4.25}$$

which indeed does *not* vanish. (In the last step I already anticipated the transition to the continuum case (volume $\Omega \to \infty$).) It is easily seen that this term describes the *acceleration* of the particles in the external field.

4.2.2 Memory Term

The memory term will be calculated to lowest (i.e., second) order in the coupling constant α. I start from the decomposition

$$\mathcal{L} = \mathcal{L}^{(E)} + \alpha\mathcal{V} \tag{4.26}$$

where the single-particle contribution $\mathcal{L}^{(E)}$ (which includes the external electric field) does not mix relevant and irrelevant degrees of freedom:

$$[\mathcal{L}^{(E)}, \mathcal{P}(t)] = 0 \quad . \tag{4.27}$$

Hence I can take over the general result (3.30) for second order perturbation theory, but with $\mathcal{U}^{(E)}$ instead of $\mathcal{U}^{(0)}$. Just as in the case of a resonance decay, formula (3.30) can be simplified further: (i) since $\mathcal{PVP} = 0$, all the $\mathcal{Q}$'s can be omitted; (ii) one may absorb $\mathcal{U}^{(E)}$ into $\mathcal{V}|N(\vec{p})\rangle\!\rangle$ by changing the latter into interaction picture operators. Here the relevant observable $N(\vec{p})$ is *not* invariant under $\mathcal{U}^{(E)}$ and hence it, too, must be replaced by the interaction picture operator $N^I(\vec{p};\tau)$. Substituting $\tau = t - t'$ yields the memory term

$$\dot{n}^{\mathrm{mem}}(\vec{p},t) = -\alpha^2 \int_0^{t-t_0} \mathrm{d}\tau\, \langle\!\langle \mathrm{rel}(t-\tau)|\mathcal{V}\mathcal{V}^I(\tau)|N^I(\vec{p};\tau)\rangle\!\rangle \quad . \tag{4.28}$$

Note that this result might have been obtained directly from (3.47) with the substitution $|i\rangle\!\rangle \rightarrow |N^I(\vec{p};\tau)\rangle\!\rangle$. Using the definition of the super-operator $\mathcal{V}$,

$$\begin{aligned}\dot{n}^{\mathrm{mem}}(\vec{p},t) &= -\frac{\alpha^2}{\hbar^2}\int_0^{t-t_0}\mathrm{d}\tau\,\langle\!\langle \mathrm{rel}(t-\tau)|[V,[V^I(\tau),N^I(\vec{p};\tau)]]\rangle\!\rangle \\ &= -\frac{\alpha^2}{16\hbar^2}\int_0^{t-t_0}\mathrm{d}\tau\sum_{abcd}\sum_{efgh}\langle\!\langle \mathrm{rel}(t-\tau)|[V_{hg|fe},[V^I_{dc|ba}(\tau),N^I(\vec{p};\tau)]]\rangle\!\rangle\end{aligned} \tag{4.29}$$

Whenever the second scattering operator $V_{dc|ba}$ and $N(\vec{p})$ commute then so do the corresponding operators in the interaction picture; in which case the above expectation value $\langle\!\langle \mathrm{rel}|\ldots\rangle\!\rangle$ vanishes. The same is true if $N(\vec{p})$ commutes with the first scattering operator $V_{hg|fe}$, because then $\langle\!\langle \mathrm{rel}|[V_{hg|fe},\ldots]\rangle\!\rangle \propto \langle\!\langle [V_{hg|fe},\rho_{\mathrm{rel}}]|\ldots\rangle\!\rangle = 0$. Hence in the above sum there is no contribution from any scattering operator that commutes with $N(\vec{p})$. Taking into account (4.20), we may effectively set

$$V_{lk|ji} := 0 \quad \text{if} \quad l=j,\ l=i,\ k=j,\ \text{or}\ k=i \quad . \tag{4.30}$$

In other words, forward scattering does not contribute to the evolution of the occupation numbers $n(\vec{p},t)$. (In general, this statement is true only in second order perturbation theory.)

In order to evaluate the expression (4.29) for the memory term I have to proceed in several steps:

- Translating (4.19) into the interaction picture, one finds

$$[V^I_{dc|ba}(\tau),N^I(\vec{p};\tau)] = (\delta_{\vec{p}_a\vec{p}}+\delta_{\vec{p}_b\vec{p}}-\delta_{\vec{p}_c\vec{p}}-\delta_{\vec{p}_d\vec{p}})V^I_{dc|ba}(\tau) \quad . \tag{4.31}$$

  One is thus left with the task of calculating $\langle\!\langle \mathrm{rel}|[V_{hg|fe},V^I_{dc|ba}(\tau)]\rangle\!\rangle$. (For brevity, I omit the time argument $(t-\tau)$ of $\langle\!\langle \mathrm{rel}|$ whenever it is not relevant to the discussion.)

- Due to the symmetries (4.16) of $V_{lk|ji}$,

$$\langle\!\langle \mathrm{rel}|[V_{hg|fe},V^I_{dc|ba}(\tau)]\rangle\!\rangle = \langle\!\langle \mathrm{rel}|V_{hg|fe}V^I_{dc|ba}(\tau)\rangle\!\rangle - (ab\leftrightarrow cd, ef\leftrightarrow gh)^* \quad . \tag{4.32}$$

  Since the relevant part of the state has the so-called Hartree form (cf. appendix C), and the $V_{lk|ji}$'s are proportional to products of field operators a, $a^\dagger$, the right-hand side can be evaluated using Wick's theorem. Considering (4.30), the right-hand side is non-zero only if both

  ▷ $\{|h\rangle,|g\rangle\} = \{U^{(E)}(-\tau)|a\rangle, U^{(E)}(-\tau)|b\rangle\}$ and
  ▷ $\{|f\rangle,|e\rangle\} = \{U^{(E)}(-\tau)|c\rangle, U^{(E)}(-\tau)|d\rangle\}$.

Given $|a\rangle, |b\rangle, |c\rangle$ and $|d\rangle$, there are thus only four possible choices for the states $|e\rangle, |f\rangle, |g\rangle$ and $|h\rangle$. Even better, again due to the symmetries (4.16), all four choices yield the same result.

- The sum over $\{e,f,g,h\}$ can now be performed, leading to

$$\begin{aligned}\sum_{efgh} \langle\!\langle \mathrm{rel}|V_{hg|fe} V^I_{dc|ba}(\tau)\rangle\!\rangle &= 4\langle ab|V^I(-\tau)|cd\rangle_\pm \langle dc|V|ba\rangle_\pm \times \\ &\quad \times \langle\!\langle \mathrm{rel}|a_a^{\dagger I}(\tau) a_a^I(\tau)\rangle\!\rangle \langle\!\langle \mathrm{rel}|a_b^{\dagger I}(\tau) a_b^I(\tau)\rangle\!\rangle \times \\ &\quad \times \langle\!\langle \mathrm{rel}|a_c^{I}(\tau) a_c^{\dagger I}(\tau)\rangle\!\rangle \langle\!\langle \mathrm{rel}|a_d^{I}(\tau) a_d^{\dagger I}(\tau)\rangle\!\rangle \quad . \end{aligned} \tag{4.33}$$

The factor 4 represents the four equivalent combinations discussed above.

- As the interaction V (and hence V^I) is Hermitian, the pre-factor

$$\begin{aligned} f_{dc|ba}(\tau) &:= \langle ab|V^I(-\tau)|cd\rangle_\pm \langle dc|V|ba\rangle_\pm \\ &= \langle dc|V|ba\rangle_\pm \langle dc|V^I(-\tau)|ba\rangle^*_\pm \end{aligned} \tag{4.34}$$

has the property

$$f^*_{ab|cd}(\tau) = f_{dc|ba}(\tau) \quad . \tag{4.35}$$

In addition, it is invariant under the interchanges $a \leftrightarrow b$ and $c \leftrightarrow d$.

- For the expectation values $\langle\!\langle \ldots | \ldots \rangle\!\rangle$ one obtains, using the definition of the relevant part,

$$\begin{aligned} \langle\!\langle \mathrm{rel}(t-\tau)|a_i^{\dagger I}(\tau) a_i^I(\tau)\rangle\!\rangle &= \frac{1}{2s+1} \langle\!\langle \mathrm{rel}(t-\tau)|N^I(\vec{p}_i;\tau)\rangle\!\rangle \\ &= \frac{1}{2s+1} n(\vec{p}_i - q\vec{E}\tau, t-\tau) \end{aligned} \tag{4.36}$$

and

$$\langle\!\langle \mathrm{rel}(t-\tau)|a_i^{I}(\tau) a_i^{\dagger I}(\tau)\rangle\!\rangle = 1 \pm \frac{1}{2s+1} n(\vec{p}_i - q\vec{E}\tau, t-\tau) \quad . \tag{4.37}$$

- The product $\prod_{abcd}\langle\!\langle \ldots | \ldots \rangle\!\rangle$ of four expectation values which appears in (4.33), is certainly not invariant under the interchange $(ab \leftrightarrow cd)$. One defines an antisymmetrized expression

$$\begin{aligned} C[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a](t,\tau) &:= n(\vec{p}_a - q\vec{E}\tau, t-\tau) n(\vec{p}_b - q\vec{E}\tau, t-\tau) \times \\ &\quad \times \left[1 \pm \frac{1}{2s+1} n(\vec{p}_c - q\vec{E}\tau, t-\tau)\right] \times \\ &\quad \times \left[1 \pm \frac{1}{2s+1} n(\vec{p}_d - q\vec{E}\tau, t-\tau)\right] - \\ &\quad -(ab \leftrightarrow cd) \quad . \end{aligned} \tag{4.38}$$

The factors $[1 \pm n/(2s+1)]$ will give rise to Bose enhancement or Pauli blocking, respectively.

- Starting from (4.32) and collecting all subsequent results and definitions, we find:

$$\sum_{efgh} \langle\!\langle \mathrm{rel}(t-\tau)|[V_{hg|fe}, V^I_{dc|ba}(\tau)]\rangle\!\rangle = 4 f_{dc|ba}(\tau) \cdot \frac{1}{(2s+1)^2} C[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a](t,\tau) \tag{4.39}$$

- In order to obtain the integrand in (4.29), this expression has yet to be multiplied by $(\delta_{\vec{p}_a\vec{p}} + \delta_{\vec{p}_b\vec{p}} - \delta_{\vec{p}_c\vec{p}} - \delta_{\vec{p}_d\vec{p}})$ and then summed over $\{a,b,c,d\}$. The only factor which depends on the spin components is $f_{dc|ba}(\tau)$; hence the spin summation may be done immediately, yielding the 'unpolarized' pre-factor

$$f[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a](\tau) := \frac{1}{(2s+1)^2} \sum_{m_a m_b} \sum_{m_c m_d} f_{dc|ba}(\tau) \quad . \tag{4.40}$$

- Due to the symmetry $\vec{p}_a \leftrightarrow \vec{p}_b$ in both $f[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a]$ and $C[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a]$, the term proportional to $\delta_{\vec{p}_a\vec{p}}$ and the term involving $\delta_{\vec{p}_b\vec{p}}$ will yield identical contributions to the sum $\sum_{\vec{p}_a\vec{p}_b\vec{p}_c\vec{p}_d}$; likewise for $\delta_{\vec{p}_c\vec{p}}$ and $\delta_{\vec{p}_d\vec{p}}$. Moreover, using the antisymmetry

$$C[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a] = -C[\vec{p}_a\vec{p}_b|\vec{p}_c\vec{p}_d] \tag{4.41}$$

and the Hermiticity

$$f[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a] = f[\vec{p}_a\vec{p}_b|\vec{p}_c\vec{p}_d]^* \quad , \tag{4.42}$$

all four terms can be summarized as

$$\mathrm{Integrand} = -16\delta_{\vec{p}_d\vec{p}} \mathrm{Re}\,\{f[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a](\tau)\}\, C[\vec{p}_d\vec{p}_c|\vec{p}_b\vec{p}_a](t,\tau) \quad . \tag{4.43}$$

- Relabelling the indices, we finally obtain

$$\begin{aligned} \dot{n}^{\mathrm{mem}}(\vec{p},t) &= \frac{\alpha^2}{\hbar^2}\mathrm{Re}\int_0^{t-t_0} \mathrm{d}\tau \sum_{\vec{p}_1\vec{p}_2\vec{p}_3} f[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau)\cdot C[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](t,\tau) \\ &\equiv \dot{n}^{\mathrm{coll}}(\vec{p},t) \quad . \end{aligned} \tag{4.44}$$

This expression has the general structure of a *collision term* as encountered in the Boltzmann equation; with the important difference, however, that it still involves an *integration over the past history.*

We have yet to unravel the meaning of the pre-factor f. Its definition involves a matrix element $\langle\ldots|V^I(-\tau)|\ldots\rangle_\pm$, which — using the evolution equation (4.10) — can be expressed in terms of an ordinary matrix element $\langle\ldots|V|\ldots\rangle_\pm$. In fact, defining

$$\begin{aligned} v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau) \quad &:= \quad \frac{1}{(2s+1)^2}\sum_{m_1m_2}\sum_{m_3m}\langle\vec{p},m;\vec{p}_3,m_3|V|\vec{p}_2,m_2;\vec{p}_1,m_1\rangle_\pm \times \\ &\times\langle\vec{p}-q\vec{E}\tau,m;\vec{p}_3-q\vec{E}\tau,m_3|V|\vec{p}_2-q\vec{E}\tau,m_2;\vec{p}_1-q\vec{E}\tau,m_1\rangle^*_\pm \end{aligned} \tag{4.45}$$

(which, in general, need not be real for arbitrary τ), defining

$$\delta\omega[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau') := \omega(\vec{p}-q\vec{E}\tau')+\omega(\vec{p}_3-q\vec{E}\tau')-\omega(\vec{p}_2-q\vec{E}\tau')-\omega(\vec{p}_1-q\vec{E}\tau') \tag{4.46}$$

and inserting an extra Kronecker symbol to make the conservation of momentum explicit, one finds:

$$f[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau) = \exp\left[i\int_0^\tau \mathrm{d}\tau'\,\delta\omega[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau')\right]\,v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau)\,\delta_{\vec{p}+\vec{p}_3,\vec{p}_1+\vec{p}_2} \tag{4.47}$$

Physically, $\delta\omega(\tau')$ is the net 'energy gain' in a collision $1+2\to 3+X$. Of course, it vanishes if the collision conserves energy. Energy conservation, however, has not yet been established. Therefore the phase factor $\exp[\ldots\delta\omega\ldots]$ may *not* be neglected. In fact, this phase factor will play a crucial role in the analysis of time scales — just as it did in the case of the resonance decay.

4.3 Time Scale Analysis

The acceleration term in the Boltzmann equation describes only a linear shift in momentum space and therefore is not associated with a characteristic time scale. The collision term, on the other hand, does exhibit two very different time scales: the memory time τ_{mem}, and the relaxation time τ_{rel}. They will be investigated separately in the following subsections. The purpose of these investigations is not an exhaustive review of all possible cases, but rather an illustration of the general method and a rough order of magnitude estimate for the time scales. I feel therefore justified in making some simplifying assumptions:

- During a time interval of length τ_{mem} the external field changes the momenta of particles only by an insignificant amount; in the collision term one may thus set $\vec{E}=0$. As a consequence, v^2 becomes time-independent:

$$v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] = \frac{1}{(2s+1)^2}\sum_{m_1m_2}\sum_{m_3m}|\langle\vec{p},m;\vec{p}_3,m_3|V|\vec{p}_2,m_2;\vec{p}_1,m_1\rangle_\pm|^2 \quad . \tag{4.48}$$

- The dynamics is nonrelativistic; hence

$$\delta\omega[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] = \frac{1}{2m\hbar}({\vec{p}}^{\,2} + {\vec{p}_3}^{\,2} - {\vec{p}_2}^{\,2} - {\vec{p}_1}^{\,2}) \quad . \tag{4.49}$$

The pre-factor f in the collision term has now a very simple time dependence:

$$f[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](\tau) = \exp(i\delta\omega\tau)\, v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1]\, \delta_{\vec{p}+\vec{p}_3,\vec{p}_1+\vec{p}_2} \quad . \tag{4.50}$$

- The system is dilute; one may thus neglect the enhancement/blocking factors $[1 \pm n/(2s+1)]$.

- The system is close to its equilibrium configuration $\{n^{\mathrm{eq}}(\vec{p})\}$, i.e.,

$$n(\vec{p},t) = n^{\mathrm{eq}}(\vec{p}) + \delta n(\vec{p},t) \tag{4.51}$$

where the deviations δn are small. More stringently, I assume that only one particular occupation number $n(\vec{p}_0,t)$ is perturbed from its equilibrium value, while all the others remain unperturbed:

$$\delta n(\vec{p},t) \begin{cases} \neq 0 & \text{if } \vec{p} = \vec{p}_0 \\ = 0 & \text{otherwise} \end{cases} \quad . \tag{4.52}$$

In the collision term the equilibrium distributions yield (by definition) $C_{\mathrm{eq}} = 0$, and only the perturbation remains. For $\vec{p} = \vec{p}_0$,

$$\delta C[\vec{p}_0\vec{p}_3|\vec{p}_2\vec{p}_1](t,\tau) = -n^{\mathrm{eq}}(\vec{p}_3)\, \delta n(\vec{p}_0, t-\tau) \quad . \tag{4.53}$$

(Variations of $n(\vec{p}_1)$ and $n(\vec{p}_2)$ do not contribute because if, for example, $\vec{p}_1 = \vec{p}_0$ then $\vec{p}_2 = \vec{p}_3$ due to momentum conservation, and hence $C = C[\vec{p}_0\vec{p}_3|\vec{p}_3\vec{p}_0] = 0$.) The collision term then describes the relaxation of $n(\vec{p}_0,t)$ towards equilibrium:

$$\begin{aligned} \dot{\delta n}^{\mathrm{coll}}(\vec{p}_0,t) \quad = \quad & -\frac{\alpha^2}{\hbar^2}\mathrm{Re}\int_0^{t-t_0} \mathrm{d}\tau \sum_{\vec{p}_1\vec{p}_2\vec{p}_3} \exp(i\delta\omega\tau)\, v^2[\vec{p}_0\vec{p}_3|\vec{p}_2\vec{p}_1] \times \\ & \times \delta_{\vec{p}_0+\vec{p}_3,\vec{p}_1+\vec{p}_2}\, n^{\mathrm{eq}}(\vec{p}_3)\, \delta n(\vec{p}_0, t-\tau) \quad . \end{aligned} \tag{4.54}$$

From this equation one can extract both the *memory time* $\tau_{\mathrm{mem}}(\vec{p}_0)$ and the *relaxation time* $\tau_{\mathrm{rel}}(\vec{p}_0)$, associated with the occupation number $n(\vec{p}_0,t)$. Strictly speaking, there is a different set of time scales $\{\tau_{\mathrm{mem}}(\vec{p}), \tau_{\mathrm{rel}}(\vec{p})\}$ for each observable $n(\vec{p},t)$. I will be content, however, with performing the analysis in the special case $\vec{p}_0 = 0$. Other memory and relaxation times can be found with the same procedure. In fact, it can be shown (but will not be shown here) that any $\tau_{\mathrm{mem}}(\vec{p})$ can never be greater than $\tau_{\mathrm{mem}}(0)$ [14]. Hence $\tau_{\mathrm{mem}}(0)$ is actually an upper limit for the various memory times, and it is fair to regard $\tau_{\rm mem}(0)$ as *the* memory time of the system: $\tau_{\rm mem}(0) \equiv \tau_{\rm mem}$. Likewise, the relaxation time $\tau_{\rm rel}(0)$ may be regarded as representative. And although, by definition, $\tau_{\rm mem}$ and $\tau_{\rm rel}$ only apply to small deviations from equilibrium, they are nevertheless useful measures for the typical time scales on which the relevant and irrelevant degrees of freedom evolve, even in situations further removed from equilibrium.

4.3.1 Memory Time

Even very close to equilibrium the system evolves rapidly on the microscopic level; it is this *microscopic* motion which manifests itself in a non-zero memory time. The memory time must be identified *before* taking the Markovian limit.

As we have learned from the example of the resonance decay, the memory time will generally depend on the *form* (but not the strength) of the interaction. If one assumes that the interaction has a certain range η, then a generic choice for the two-body potential is the Gaussian potential

$$V(r) \propto \exp(-r^2/\eta^2) \quad . \tag{4.55}$$

Neglecting the effects of (anti-)symmetrization of the matrix elements, this leads to

$$v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] \propto \exp(-q^2\eta^2/2) \quad ; \tag{4.56}$$

where

$$\hbar\vec{q} := \vec{p} - \vec{p}_1 = \vec{p}_2 - \vec{p}_3 \tag{4.57}$$

is the wave vector associated with the momentum transfer in a collision. Furthermore, defining

$$\vec{v} := \frac{1}{m}(\vec{p} - \vec{p}_2) = \frac{1}{m}(\vec{p}_1 - \vec{p}_3) \tag{4.58}$$

one may write

$$\delta\omega = \vec{q} \cdot \vec{v} \quad . \tag{4.59}$$

To proceed further, it is also necessary to make some assumption about the shape of the equilibrium distribution. Again, I neglect the effect of (anti-)symmetrization and choose a Maxwell-Boltzmann (rather than a Bose or Fermi) distribution with inverse temperature β:

$$n^{\rm eq}(\vec{p}_3, t) \propto \exp(-\beta p_3^2/2m) \quad . \tag{4.60}$$

Next I have to evaluate the right-hand side of the relaxation equation (4.54). In order to do that, I (i) assume that the volume Ω is large, permitting me to take the continuum limit (cf. appendix A):

$$\sum_{\vec{p}_i} \to \frac{\Omega}{h^3}\int {\rm d}^3p_i \quad \text{and} \quad \delta_{\vec{p}_i\vec{p}_j} \to \frac{h^3}{\Omega}\delta^3(\vec{p}_i - \vec{p}_j) \quad ; \tag{4.61}$$

(ii) absorb the δ^3-function into the integration over $\vec{p}_3$; (iii) substitute $(\vec{p}_1, \vec{p}_2) \to (\vec{q}, \vec{v})$; and (iv) use $\vec{p} = \vec{p}_0 = 0$. I obtain (up to a normalization factor)

$$\begin{aligned} \dot{\delta n}^{\mathrm{coll}}(0,t) \quad \propto \quad & \mathrm{Re} \int_0^{t-t_0} \mathrm{d}\tau \int \mathrm{d}^3 v \int \mathrm{d}^3 q \, \exp \left[i\vec{q}\cdot\vec{v}\tau - \frac{1}{2} q^2 \eta^2 - \right. \\ & \left. - \beta \left(\frac{\hbar^2 q^2}{2m} + \frac{m v^2}{2} + \hbar \vec{q}\cdot\vec{v} \right) \right] \delta n(0, t-\tau) \quad . \qquad (4.62) \end{aligned}$$

The integration over $\vec{v}$ amounts to Fourier transforming a Gaussian with width $(\beta m)^{-1/2}$, yielding a Gaussian distribution in $(\tau + i\hbar\beta)\vec{q}$ with width $(\beta m)^{+1/2}$. One is left with

$$\begin{aligned} \dot{\delta n}^{\mathrm{coll}}(0,t) & \propto \mathrm{Re} \int_0^{t-t_0} \mathrm{d}\tau \int \mathrm{d}^3 q \exp \left[-\frac{q^2}{2} \left(\frac{(\tau + i\hbar\beta)^2}{\beta m} + \eta^2 + \frac{\hbar^2 \beta}{m} \right) \right] \delta n(0, t-\tau) \\ & \propto \mathrm{Re} \int_0^{t-t_0} \mathrm{d}\tau \, (\tau^2 + 2i\hbar\beta\tau + \eta^2 \beta m)^{-3/2} \; \delta n(0, t-\tau) \quad . \qquad (4.63) \end{aligned}$$

Any normalization factors have been ignored, since I am only interested in the form of the integrand's τ-dependence. Clearly, the integrand exhibits two characteristic time scales, $(\eta\sqrt{\beta m})$ and $(\hbar\beta)$. Times τ which are greater than both of these characteristic scales yield only a negligible contribution to the integral. In other words, only those times τ contribute significantly which are smaller than either $(\eta\sqrt{\beta m})$ or $(\hbar\beta)$. This finding may be summarized in a compact fashion by setting

$$\tau_{\mathrm{mem}} = \sqrt{\eta^2 \beta m + (\hbar\beta)^2} \quad . \qquad (4.64)$$

The two contributions to the memory time, $(\eta\sqrt{\beta m})$ and $(\hbar\beta)$, have very different origins:

- The time $\tau_{\mathrm{mem}}^{\mathrm{class}} := \eta\sqrt{\beta m}$ is independent of $\hbar$ and therefore of entirely classical origin. Taking into account that $\beta \propto (m\langle v^2 \rangle)^{-1}$, one finds that

  $$\tau_{\mathrm{mem}}^{\mathrm{class}} \propto \frac{\eta}{\sqrt{\langle v^2 \rangle}} \qquad (4.65)$$

  is just the average time needed for a particle to pass through an interaction range. Only during that time may correlations between colliding particles not be neglected. The classical memory time depends both on the form of the interaction (through η) and on the particle distribution (through β).

- The time $\tau_{\mathrm{mem}}^{\mathrm{quant}} := \hbar\beta$ is proportional to $\hbar$ and therefore of purely quantum mechanical origin. It is obviously a remnant of the coherent superposition of individual $2 \to 2$ scattering processes. More precisely, it describes the average temporal extent of a quantum mechanical scattering process associated (via the time-energy uncertainty relation) with the characteristic energy scale β^{-1} of the system. For shorter times $\tau < \hbar\beta$ energy conservation is not yet established, the colliding particles are still off mass-shell, and hence any subsequent collision would not be independent of the first; in short, for $\tau < \hbar\beta$ a scattering process cannot be regarded as completed. The quantum mechanical memory time (also called 'off-shell' time) is independent of the interaction and depends only on the particle distribution (through β).

Both the classical and the quantum mechanical memory time decrease with increasing temperature. However, the temperature-dependence of the two memory times differs: $\sqrt{\beta}$ vs. β. There is a 'critical' temperature T_c at which both memory times are of equal importance:

$$kT_c = \frac{(\hbar c)^2}{mc^2\eta^2} \tag{4.66}$$

(with $\hbar c \approx 200$ MeV fm). For lower temperatures, $T < T_c$, the quantum mechanical memory time dominates; while for higher temperatures, $T > T_c$, the classical memory time dominates. Example: For a gas of interacting nucleons ($mc^2 \approx 1$ GeV, $\eta \approx 1$ fm), the 'critical' temperature is $kT_c \approx 40$ MeV.

Let me finally mention that the above result for the memory time agrees with an earlier analysis by Danielewicz who, instead of the projection method, employed Green's function techniques [14].

4.3.2 Relaxation Time

Provided the memory time is sufficiently small, one can (i) take the Markovian limit, $\delta n(\vec{p}_0, t-\tau) \to \delta n(\vec{p}_0, t)$, and (ii) do the quasistationary approximation, $(t - t_0) \to \infty$. The τ-integration in the relaxation equation (4.54) can then be performed immediately, yielding

$$\mathrm{Re} \int_0^\infty \mathrm{d}\tau \, \exp(i\delta\omega\tau) = \frac{1}{2}\delta(\delta\omega) \quad . \tag{4.67}$$

The above approximations thus guarantee energy conservation in each individual collision.

According to Fermi's golden rule [86], the transition rate for scattering $(\vec{p}_3, \vec{p}) \to (\vec{p}_1, \vec{p}_2)$ is given by

$$r[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] = \frac{\alpha^2}{\hbar^2}\delta(\delta\omega)\, v^2[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1]\, \delta_{\vec{p}+\vec{p}_3,\vec{p}_1+\vec{p}_2} \quad . \tag{4.68}$$

Taking the sum over all final states $(\vec{p}_1, \vec{p}_2)$, dividing by the incident flux $|\vec{v}_3 - \vec{v}|/\Omega$, and multiplying by an extra factor $1/2$ to account for the indistinguishability of the colliding particles, yield the total cross section

$$\sigma_{\mathrm{tot}}[\vec{p},\vec{p}_3] = \frac{\Omega}{|\vec{v}_3 - \vec{v}|} \cdot \frac{1}{2} \sum_{\vec{p}_1,\vec{p}_2} r[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] \quad . \tag{4.69}$$

Using this expression for the total cross section, the relaxation equation takes on the simple form

$$\dot{\delta n}^{\mathrm{coll}}(\vec{p}_0,t) = -\frac{1}{\Omega}\left(\sum_{\vec{p}_3} |\vec{v}_3 - \vec{v}_0|\,\sigma_{\mathrm{tot}}[\vec{p}_0,\vec{p}_3]\,n^{\mathrm{eq}}(\vec{p}_3)\right)\delta n(\vec{p}_0,t) \quad . \tag{4.70}$$

It has the structure of an exponential decay law. For $\vec{p}_0 = \vec{v}_0 = 0$, the decay width is given by

$$\begin{aligned} \Gamma &= \frac{1}{\Omega}\sum_{\vec{p}'} |\vec{v}'|\,\sigma_{\mathrm{tot}}[0,\vec{p}']\,n^{\mathrm{eq}}(\vec{p}') \\ &= n\langle v\sigma\rangle \quad , \end{aligned} \tag{4.71}$$

where $n := N/\Omega$ denotes the particle density and $\langle\ldots\rangle$ the equilibrium average. Associated with this decay width is the relaxation time

$$\tau_{\mathrm{rel}} = \frac{1}{n\langle v\sigma\rangle} \quad . \tag{4.72}$$

The relaxation time is intimately connected with the mean free path $\lambda_f = 1/n\sigma$; in fact, it is nothing but the *average time between two collisions.* This result agrees completely with our classical intuition. With hindsight, we may now conclude that the Markovian limit was justified provided the average time between collisions is much larger than the temporal extent of each individual scattering process. The Markovian limit becomes exact as $\alpha \to 0$ (and hence $\sigma \to 0$) or $n \to 0$; i.e., for weakly interacting or very dilute systems. In contrast, the quality of the Markovian limit does not generally improve with increasing temperature, since the average velocity drops out of the ratio $\tau_{\mathrm{mem}}^{\mathrm{class}}/\tau_{\mathrm{rel}}$. (It might improve nevertheless if the cross section decreases with increasing temperature.)

4.4 Limit: The Classical Boltzmann Equation

While the preceding time scale analysis was confined to systems near equilibrium, I return now to the general case of arbitrary momentum distributions. As in the previous section, however, I do assume that the Markovian and quasistationary limits can be taken. Then in the collision term (4.44) the τ-integration can be performed, and — as before — the result can be expressed in terms of the transition rate:

$$\dot{n}^{\mathrm{coll}}(\vec{p},t) = \frac{1}{2}\sum_{\vec{p}_1\vec{p}_2\vec{p}_3} r[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1]\,C[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](t,0) \quad . \tag{4.73}$$

The transition rate, in turn, may be related to the differential cross section. To see that, let us consider a collision in the center of mass frame: $(\vec{P}, -\vec{P}) \to (\vec{Q}, -\vec{Q})$. The differential cross section for this process is given by

$$(\mathrm{d}\sigma)_{\mathrm{cm}} = \frac{\Omega m}{2|\vec{P}|} \sum_{\vec{Q}} r[\vec{P}, -\vec{P}|\vec{Q}, -\vec{Q}] \quad . \tag{4.74}$$

Using the nonrelativistic formula $\epsilon_Q = Q^2/2m$ for the kinetic energy, one may go to the continuum limit via

$$\sum_{\vec{Q}} \to \frac{\Omega}{h^3} \int \mathrm{d}^3 Q = \frac{\Omega}{(2\pi\hbar)^3} \cdot \frac{m}{2} \int \mathrm{d}\omega \int \mathrm{d}(\epsilon_Q + \epsilon_{-Q})\, |\vec{Q}| \quad . \tag{4.75}$$

(Here ω denotes the solid angle, *not* the frequency.) Due to energy conservation, $|\vec{Q}| = |\vec{P}|$; the momenta thus cancel, leaving

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{cm}} = \frac{\Omega^2}{(2\pi\hbar)^3} \left(\frac{m}{2}\right)^2 \int \mathrm{d}(\epsilon_Q + \epsilon_{-Q})\, r[\vec{P}, -\vec{P}|\vec{Q}, -\vec{Q}] \quad . \tag{4.76}$$

Conversely,

$$r[\vec{P}, -\vec{P}|\vec{Q}, -\vec{Q}] = \frac{(2\pi\hbar)^3}{\Omega^2} \left(\frac{2}{m}\right)^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{cm}} \delta(\epsilon_Q + \epsilon_{-Q} - \epsilon_P - \epsilon_{-P}) \quad . \tag{4.77}$$

Since both the transition rate and the δ-function (but *not* the cross section!) are invariant under Galileo transformations, this relation may be boosted to an arbitrary frame, yielding

$$r[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1] = \frac{(2\pi\hbar)^3}{\Omega^2} \left(\frac{2}{m}\right)^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{cm}} \delta\left(\sum \epsilon_i\right) \quad . \tag{4.78}$$

It is important that the differential cross section is still evaluated in the center of mass frame. Furthermore, this boosting 'trick' is only possible in the nonrelativistic case.

In the continuum limit, the occupation numbers are normalized to

$$\int \mathrm{d}^3 p\, n(\vec{p}, t) = (2\pi\hbar)^3 \frac{N}{\Omega} \quad . \tag{4.79}$$

They are usually replaced by another distribution function

$$f(\vec{p}, t) := \frac{1}{(2\pi\hbar)^3} n(\vec{p}, t) \tag{4.80}$$

such that

$$\int \mathrm{d}^3p\, f(\vec{p},t) = \frac{N}{\Omega} \quad . \tag{4.81}$$

Defining

$$\tilde{C}[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](t) := f(\vec{p}_1,t)f(\vec{p}_2,t) - f(\vec{p}_3,t)f(\vec{p},t) \quad , \tag{4.82}$$

the collision term now reads

$$\begin{aligned} I[f] &:= \dot{f}^{\mathrm{coll}}(\vec{p},t) \\ &= \frac{1}{2}\int \mathrm{d}^3p_1 \int \mathrm{d}^3p_2 \int \mathrm{d}^3p_3 \left(\frac{2}{m}\right)^2 \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{cm}} \times \\ &\quad \times \delta\left(\sum \epsilon_i\right) \delta^3\left(\sum \vec{p}_i\right) \cdot \tilde{C}[\vec{p}\vec{p}_3|\vec{p}_2\vec{p}_1](t) \quad . \end{aligned} \tag{4.83}$$

This is indeed the collision term of the classical Boltzmann equation [61].

So far the distribution $f(\vec{p},t)$ has been defined only on momentum space ($\vec{p}$), but not on phase space ($\vec{r},\vec{p}$). Therefore it cannot describe spatial variations of the particle density. This is not surprising, since $\vec{r}$ and $\vec{p}$ are non-commuting observables; hence in a quantum mechanical description the concept of a 'point in phase space' does not make sense. However, if the spatial variation of the particle density is sufficiently small (the exact mathematical conditions for 'sufficiency' being left to further analysis [14]), then space can be divided into macroscopic cells, each of which may be regarded as spatially homogeneous. (The characteristic length scale which separates the macroscopic from the microscopic domain is furnished by the range of the interaction, η, and the typical de Broglie wavelength λ_{dB} associated with the average momentum of the particles.) Each individual macroscopic cell i, located at $\vec{r}_i$, is described by a momentum distribution $f(\vec{r}_i,\vec{p},t)$. The Boltzmann equation which governs the evolution of the *entire* system is then simply the quantum Boltzmann equation for each individual cell, modified by a 'drift term' $\vec{v}\cdot\nabla_{\vec{r}} f$ which describes particle exchange between cells:

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}} + q\vec{E}\cdot\nabla_{\vec{p}}\right) f(\vec{r},\vec{p},t) = I[f] \quad . \tag{4.84}$$

This is the *classical Boltzmann equation.**

To conclude, let me list all approximations which led to this result:

- lowest (i.e., second) order perturbation theory
- dilute gas approximation (neglecting enhancement or blocking factors)
- Markovian limit

*An alternative derivation of the classical Boltzmann equation, frequently encountered in the literature, involves taking the classical limit ($\hbar \to 0$) of the equation of motion for the Wigner function [87].

- quasistationary approximation
- spatial homogeneity on the microscopic scale (i.e., η and λ_{dB}); macroscopic inhomogeneities 'sufficiently weak'.

4.5 Mixtures

Often various types $\{i\}$ of particles (e.g., electrons and positrons) are mixed. They interact both with each other and amongst themselves, with respective transition rates r_{ij}. In general, this leads to an entire set of coupled Boltzmann equations. In the case of an e^+e^- mixture with respective occupation numbers n_+ and n_- one obtains (schematically):

$$\left[\frac{\partial}{\partial t} + e\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{E} \cdot \nabla_{\vec{p}}\right] \begin{pmatrix} n_+(\vec{p},t) \\ n_-(\vec{p},t) \end{pmatrix} = \sum_{\vec{p}_1\vec{p}_2\vec{p}_3} \begin{pmatrix} \frac{1}{2}r_{++}C_{++} + r_{+-}C_{+-} \\ r_{-+}C_{-+} + \frac{1}{2}r_{--}C_{--} \end{pmatrix} \tag{4.85}$$

The factor 1/2 on the right-hand side applies only to scattering of identical particles [46]. Creation or annihilation processes have not yet been taken into account. aken into account.

Chapter 5

Particle Creation

In the presence of a strong electric field the vacuum becomes instable; it decays by emitting charged particle-antiparticle pairs, such as electrons and positrons. For the case of spin-1/2 fermions, the number of produced pairs per unit time and unit volume is given by Schwinger's famous formula

$$w = \frac{(qE)^2}{4\pi^3\hbar^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{n\pi m^2}{\hbar q E}\right) \tag{5.1}$$

(with $c = 1$) [31, 88, 89, 90, 91, 92, 93, 94, 95]. Here m denotes the fermion mass, q its electric charge, and E the magnitude of the external electric field, which is assumed to be time-independent and homogeneous. In a similar manner pair creation occurs in the presence of other fields, such as gauge fields (e.g., chromoelectric fields) or gravitational fields. Pair creation in strong chromoelectric fields may give rise to the fragmentation of chromoelectric flux tubes, a mechanism which is frequently invoked to model hadron production in high-energy reactions. Pair creation in strong gravitational fields is responsible for the Hawking radiation of black holes.

In this chapter, I will concentrate on the electric field case. This is the simplest available model for particle creation, yet it exhibits all the features which are of conceptual interest: separation of time scales, apparent irreversibility, and entropy production.

5.1 Group Theoretical Approach and the Spin-Flip Analogy

The creation of e^+e^--pairs in an external electric field is a single-particle problem. The appropriate starting point is therefore the Dirac equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H^{(E)}|\psi(t)\rangle \tag{5.2}$$

with

$$H^{(E)} = \hat{\vec{p}} \cdot \vec{\alpha} + m\beta + qA_0 \quad . \tag{5.3}$$

(For electrons, $q = -|e|$.) For a time-independent, homogeneous electric field $\vec{E}$ the potential is given by

$$A_0(\vec{r}) = -\vec{E} \cdot \vec{r} \quad . \tag{5.4}$$

Defining

$$\vec{p}(t) := \vec{p} + q\vec{E}t \quad , \tag{5.5}$$

I make the plane wave ansatz

$$|\psi(t)\rangle = |\vec{p}(t)\rangle \otimes |\chi_{\vec{p}}(t)\rangle \tag{5.6}$$

with

$$\langle \vec{r}|\vec{p}(t)\rangle = \frac{1}{\sqrt{\Omega}} \exp\left[\frac{i}{\hbar}\vec{p}(t) \cdot \vec{r}\right] \quad . \tag{5.7}$$

The remaining factor, $|\chi_{\vec{p}}(t)\rangle$, is a four-component Dirac spinor. Choosing a momentum eigenstate with *variable* momentum $\vec{p}(t)$ takes the acceleration of the particle into account. Pair creation, on the other hand, is encoded entirely in the evolution of the spinor $|\chi_{\vec{p}}(t)\rangle$.

Inserting the above ansatz into the Dirac equation yields

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\chi_{\vec{p}}(t)\rangle = [\vec{p}(t) \cdot \vec{\alpha} + m\beta]|\chi_{\vec{p}}(t)\rangle \quad . \tag{5.8}$$

This 'reduced' equation of motion exhibits a remarkable dynamical symmetry, discovered first by Perelomov [96, 97] and subsequently discussed by other authors [98]. It will be presented below.

For simplicity, one assumes that the external field is parallel to the z-axis,

$$\vec{E} = E\vec{e}_z \quad . \tag{5.9}$$

Then the transverse momentum $\vec{p}_\perp := (p_x, p_y, 0)$ is a constant of the motion; and so is the transverse energy ($c = 1$)

$$\epsilon_\perp := \sqrt{m^2 + p_\perp^2} \quad . \tag{5.10}$$

Defining the vector

$$\vec{B}[\vec{p}(t)] := (2\epsilon_\perp, 0, 2p_{||}(t)) \tag{5.11}$$

and the operators in spinor space

$$J^1_{\vec{p}_\perp} := \frac{1}{2\epsilon_\perp}(\vec{p}_\perp \cdot \vec{\alpha} + m\beta) \quad , \tag{5.12}$$

$$J^2_{\vec{p}_\perp} := \frac{i}{2\epsilon_\perp}(\vec{p}_\perp \cdot \vec{\alpha} + m\beta)\alpha_z \quad , \tag{5.13}$$

$$J^3_{\vec{p}_\perp} := \frac{1}{2}\alpha_z \quad , \tag{5.14}$$

the equation of motion may be written

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\chi_{\vec{p}}(t)\rangle = \vec{B}[\vec{p}(t)]\cdot\vec{J}_{\vec{p}_\perp}|\chi_{\vec{p}}(t)\rangle \quad . \tag{5.15}$$

The operators $\{J^i_{\vec{p}_\perp}\}$ satisfy the commutation relations

$$[J^i_{\vec{p}_\perp}, J^j_{\vec{p}_\perp}] = i\epsilon^{ijk}J^k_{\vec{p}_\perp} \tag{5.16}$$

and thus are a representation of the generators of $su(2)$. Since

$$\sum_i (J^i_{\vec{p}_\perp})^2 = \tfrac{3}{4} = \tfrac{1}{2}(\tfrac{1}{2}+1) \quad , \tag{5.17}$$

the representation corresponds to spin 1/2. Consequently, the Hamiltonian

$$H[\vec{p}(t)] := \vec{B}[\vec{p}(t)]\cdot\vec{J}_{\vec{p}_\perp} \tag{5.18}$$

is an element of a spin-1/2 representation of the Lie algebra of $SU(2)$. One concludes that *the evolution of* $|\chi_{\vec{p}}(t)\rangle$ *is formally equivalent to the motion of a spin* $1/2$ *in a time-dependent magnetic field* $\vec{B}[\vec{p}(t)]$.

The dynamical $SU(2)$ symmetry opens up the way towards a very intuitive picture of the pair creation process. First of all, the evolution of the four-component spinor $|\chi_{\vec{p}}(t)\rangle$ is essentially reduced to a two-dimensional problem. Mapping the generators of the dynamical symmetry to the Pauli matrices,

$$\{J^i_{\vec{p}_\perp}\} \to \{\tfrac{1}{2}\sigma^i\} \quad , \tag{5.19}$$

the equation of motion takes on the form

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\chi}_{\vec{p}}(t)\rangle = \tfrac{1}{2}\,\vec{B}[\vec{p}(t)]\cdot\vec{\sigma}\,|\tilde{\chi}_{\vec{p}}(t)\rangle \tag{5.20}$$

where $|\tilde{\chi}\rangle$ is now a two-component, rather than four-component, spinor. The Hamiltonian

$$\begin{aligned}\tilde{H}[\vec{p}(t)] &:= \tfrac{1}{2}\,\vec{B}[\vec{p}(t)]\cdot\vec{\sigma} \\ &= \begin{pmatrix} p_{||}(t) & \epsilon_\perp \\ \epsilon_\perp & -p_{||}(t)\end{pmatrix}\end{aligned} \tag{5.21}$$

has the eigenvalues

$$\pm\epsilon[\vec{p}(t)] := \pm\sqrt{\epsilon_\perp^2 + p_{||}(t)^2} \quad , \tag{5.22}$$

which obviously correspond to positive energy (particle) and negative energy (antiparticle) solutions of the Dirac equation. Yet at the same time — referring to the dynamical $SU(2)$ group — they correspond to 'spin aligned' and 'spin anti-aligned' states. ('Aligned' and 'anti-aligned' refer to the 'magnetic field' $\vec{B}[\vec{p}(t)]$.) Hence formally, positive energy solutions may be identified with 'spin aligned,' negative energy solutions with 'spin anti-aligned' states. Then *the creation of an* e^+e^-*-pair is formally equivalent to a spin-flip process.*

The eigenstates of the Hamiltonian are denoted by $|\xi^\pm[\vec{p}(t)]\rangle$; i.e.,

$$\tilde{H}[\vec{p}(t)]|\xi^\pm[\vec{p}(t)]\rangle = \pm\epsilon[\vec{p}(t)]|\xi^\pm[\vec{p}(t)]\rangle \quad . \tag{5.23}$$

Like the eigenvalues, they vary in time. Using the representation (5.21) of the Hamiltonian, the eigenstates can be determined as

$$|\xi^\pm[\vec{p}(t)]\rangle = \frac{1}{\sqrt{2\epsilon[\vec{p}(t)](\epsilon[\vec{p}(t)] \mp p_{||}(t))}} \begin{pmatrix} \pm\epsilon_\perp \\ \epsilon[\vec{p}(t)] \mp p_{||}(t) \end{pmatrix} \tag{5.24}$$

(up to an arbitrary phase). The explicit time-dependence of the Hamiltonian entails that these eigenstates are no longer stationary states. An energy eigenstate will generally not remain an energy eigenstate; rather, it will evolve into a superposition:

$$\tilde{U}(t_f,t_i)\begin{pmatrix} |\xi_i^+\rangle \\ |\xi_i^-\rangle \end{pmatrix} = \begin{pmatrix} \alpha_{fi} & \beta_{fi} \\ -\beta_{fi}^* & \alpha_{fi}^* \end{pmatrix}\begin{pmatrix} \mathrm{e}^{+i\phi_{fi}} & 0 \\ 0 & \mathrm{e}^{-i\phi_{fi}} \end{pmatrix}\begin{pmatrix} |\xi_f^+\rangle \\ |\xi_f^-\rangle \end{pmatrix} \tag{5.25}$$

Here I employed the shorthand notation $|\xi_{i,f}^\pm\rangle \equiv |\xi^\pm[\vec{p}(t_{i,f})]\rangle$. The operator $\tilde{U}(t_f,t_i)$ is the evolution operator associated with the Hamiltonian $\tilde{H}[\vec{p}(t)]$; it transforms an initial state at time t_i, with initial momentum $\vec{p}(t_i)$, into a final state at time t_f, with final momentum $\vec{p}(t_f)$. If the energy eigenstates were stationary (which they are not), then they would simply accumulate the dynamical phase

$$\pm\phi_{fi} := \mp\frac{1}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t'\, \epsilon[\vec{p}(t')] \tag{5.26}$$

(cf. (4.10)); in reality, however, the evolution also mixes positive and negative energy eigenstates, with respective amplitudes α_{fi} and β_{fi}. It is $|\alpha|^2 + |\beta|^2 = 1$: as it should be, the transformation matrix is an element of $SU(2)$. Physically, β_{fi} and $(-\beta_{fi}^*)$ represent the amplitudes for a spin-flip ('aligned→ anti-aligned' or 'anti-aligned→aligned'), and $|\beta_{fi}|^2$ the associated probability. In other words, $|\beta_{fi}|^2$ equals the *probability for having created an* e^+e^-*-pair with (final) momenta* $\pm\vec{p}(t_f)$ *during the time interval* $[t_i, t_f]$.

If the external field is very weak, and hence $\tilde{H}[\vec{p}(t)]$ varies very slowly, then the adiabatic theorem applies [99, 100]. In this adiabatic limit, $\beta|_{\mathrm{adiab}} = 0$ and $\alpha|_{\mathrm{adiab}} = 1$: no pair creation occurs. *Pair creation* (which requires $\beta \neq 0$) *is a non-adiabatic correction that arises at sufficiently large field strengths.*

In the general case, the amplitudes α and β are (by definition) given by

$$\begin{aligned}\alpha_{fi} &= \mathrm{e}^{-i\phi_{fi}}\langle\xi_f^+|\tilde{U}(t_f,t_i)|\xi_i^+\rangle \quad ,\\ \beta_{fi} &= \mathrm{e}^{+i\phi_{fi}}\langle\xi_f^-|\tilde{U}(t_f,t_i)|\xi_i^+\rangle \quad .\end{aligned} \tag{5.27}$$

Clearly, they depend on both the initial and the final time. Derivatives with respect to the *final* time are abbreviated as

$$\dot{\alpha}_{fi} := \frac{\partial}{\partial t_f}\alpha_{fi} \quad , \quad \dot{\beta}_{fi} := \frac{\partial}{\partial t_f}\beta_{fi} \quad . \tag{5.28}$$

These derivatives can be expressed in terms of the original amplitudes, leading to the differential equation

$$\begin{pmatrix} \dot{\alpha}_{fi} \\ \dot{\beta}_{fi} \end{pmatrix} = \begin{pmatrix} 0 & \langle\dot{\xi}_f^+|\xi_f^-\rangle\mathrm{e}^{-i2\phi_{fi}} \\ \langle\dot{\xi}_f^-|\xi_f^+\rangle\mathrm{e}^{+i2\phi_{fi}} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{fi} \\ \beta_{fi} \end{pmatrix} \tag{5.29}$$

with initial conditions

$$\alpha_{ii} = 1 \quad , \quad \beta_{ii} = 0 \quad . \tag{5.30}$$

The scalar products which occur in the above matrix can be evaluated:

$$\begin{aligned}\langle\dot{\xi}_f^-|\xi_f^+\rangle &= \frac{qE}{2}\cdot\frac{\epsilon_\perp}{\epsilon[\vec{p}(t_f)]^2} \\ &= -\langle\dot{\xi}_f^+|\xi_f^-\rangle \quad .\end{aligned} \tag{5.31}$$

In principle, one can then solve the differential equation and determine the amplitudes exactly. For our purposes, however, this will not be necessary.

In all our considerations the *physical* spin of the particle, i.e., its intrinsic angular momentum, has been irrelevant: the (physical) spin component, m, is unaffected by the external electric field.* Using the factorization (5.6), we may thus write down the complete evolution law

$$U^{(E)}(t_f,t_i)\begin{pmatrix} |i,+\rangle \\ |i,-\rangle \end{pmatrix} = \begin{pmatrix} \alpha_{fi} & \beta_{fi} \\ -\beta_{fi}^* & \alpha_{fi}^* \end{pmatrix}\begin{pmatrix} \mathrm{e}^{+i\phi_{fi}} & 0 \\ 0 & \mathrm{e}^{-i\phi_{fi}} \end{pmatrix}\begin{pmatrix} |f,+\rangle \\ |f,-\rangle \end{pmatrix} \tag{5.32}$$

with $|i,\pm\rangle \equiv |\vec{p}(t_i),m,\pm\rangle$ (and an analogous definition for $|f,\pm\rangle$).

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*The symbol m is used to denote both the mass and the spin component. Its particular meaning should become clear from the context.

5.2 Bogoliubov Transformation

The results of the previous section now have to be translated into the language of field operators. Let $\mathcal{L}^{(E)}$ be the Liouvillian and $\mathcal{U}^{(E)}$ the evolution (super-)-operator corresponding to $H^{(E)}$ and $U^{(E)}$, respectively. Furthermore, let us employ the shorthand notations $a_j \equiv a(\vec{p}(t_j), m)$ and $b_{-j} \equiv b(-\vec{p}(t_j), -m)$. Then according to the general rule

$$\mathcal{U}(t_2, t_1) a^\dagger(\psi) = a^\dagger(U(t_2, t_1)\psi) \quad , \tag{5.33}$$

equation (5.32) implies:

$$\mathcal{U}^{(E)}(t_2, t_1) \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} = \begin{pmatrix} \alpha_{21} & \beta_{21} \\ -\beta_{21}^* & \alpha_{21}^* \end{pmatrix} \begin{pmatrix} \mathrm{e}^{+i\phi_{21}} & 0 \\ 0 & \mathrm{e}^{-i\phi_{21}} \end{pmatrix} \begin{pmatrix} a_2^\dagger \\ b_{-2} \end{pmatrix} \tag{5.34}$$

(In the absence of pair creation this reduces to (4.11), with $\tau \equiv t_1 - t_2$.) Hermitian conjugation yields:

$$\mathcal{U}^{(E)}(t_2, t_1) \begin{pmatrix} a_1 \\ b_{-1}^\dagger \end{pmatrix} = \begin{pmatrix} \alpha_{21}^* & \beta_{21}^* \\ -\beta_{21} & \alpha_{21} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\phi_{21}} & 0 \\ 0 & \mathrm{e}^{+i\phi_{21}} \end{pmatrix} \begin{pmatrix} a_2 \\ b_{-2}^\dagger \end{pmatrix} \tag{5.35}$$

Pair creation, therefore, is described mathematically by a time-dependent *Bogoliubov transformation* [101].

It is important to check that the Bogoliubov transformation preserves the canonical anti-commutation relations

$$[A_i, A_j]_+ = [A_i^\dagger, A_j^\dagger]_+ = 0 \quad , \tag{5.36}$$

$$[A_i, A_j^\dagger]_+ = \delta_{ij} \quad , \tag{5.37}$$

where $i, j = 1, 2$ and A is a two-component object $(A_1, A_2) := (a(\psi), b^\dagger(\chi))$. The anti-commutation relations are only preserved by linear maps

$$A_i \to \tilde{A}_i = \Lambda_i^k A_k \tag{5.38}$$

which satisfy $\Lambda\Lambda^\dagger = 1$; i.e., $\Lambda \in U(2)$. But we already know that the transformation matrix is unitary; hence the Bogoliubov transformation is compatible with the underlying Fermi-Dirac statistics.

In fact, one can turn the tables and regard this as yet another proof of the spin-statistics theorem. If one requires that the particle statistics be such that it can accomodate the dynamical $SU(2)$ symmetry, then the particles must necessarily be fermions. On the other hand, the Klein-Gordon equation exhibits a dynamical $SU(1,1)$ symmetry [102], which is compatible with the commutation relations only if the particles are bosons.

5.3 Source Term of the Quantum Boltzmann Equation

In this section, I will incorporate pair creation into the kinetic description of e^+e^--mixtures, thereby arriving at a generalisation of the quantum Boltzmann equation. The new equation will describe

- acceleration in the external field,
- two-particle scattering (via an effective 4-fermion interaction), and
- pair creation and annihilation in the external field;

but *not*

- backreaction of the produced charges (and currents) on the external field,
- emission, absorption, or scattering of real photons.

Since we are now dealing with a mixture of particles (electrons and positrons), the **level of description** is spanned by two sets of number operators:

$$\begin{aligned} N_-(\vec{p}) &:= \sum_m a^\dagger(\vec{p},m)a(\vec{p},m) \quad , \\ N_+(\vec{p}) &:= \sum_m b^\dagger(\vec{p},m)b(\vec{p},m) \quad , \end{aligned} \tag{5.39}$$

with respective expectation values $n_-(\vec{p},t)$ and $n_+(\vec{p},t)$. In general, this will give rise to two coupled Boltzmann equations of the form (4.85), modified by a term describing pair creation. However, if the initial state is the vacuum,

$$\langle\!\langle \rho(t_0)| = \langle\!\langle 0| \quad , \tag{5.40}$$

then momentum and charge conservation dictate

$$n_+(\vec{p},t) = n_-(-\vec{p},t) \tag{5.41}$$

for all later times t. Hence for our purposes it is sufficient to consider only the evolution of, say, the electron occupation numbers $n_-(\vec{p},t)$.

Mathematically, the time-dependent Bogoliubov transformation contained in $\mathcal{U}^{(E)}$ may give rise to corrections to both the first term and the memory term of the Robertson equation. These corrections — if they exist — will be calculated and collected into one additional term in the quantum Boltzmann equation: the source term. Throughout the following investigations I will assume that particle production is weak; i.e., that $|\beta| \approx 0$. Therefore corrections which are proportional to the amplitude β will be neglected. As it turns out, this approximation amounts to neglecting all but the leading term ($n = 1$) in Schwinger's formula (5.1); which is justified for weak fields, $\hbar qE \ll m^2$ (cf. section 5.4). In contrast, derivatives of β need not be small; terms proportional to a derivative of β have to be retained.

Corresponding to the decomposition of the transformation matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ -\beta^* & 0 \end{pmatrix} \tag{5.42}$$

into its diagonal and off-diagonal parts, the single-particle evolution operator $\mathcal{U}^{(E)}$ can be split into two parts:

$$\mathcal{U}^{(E)} = \mathcal{U}^{(E)}_{\text{diag}} + \delta\mathcal{U}^{(E)} \quad . \tag{5.43}$$

Consequently, the Liouvillian

$$\mathcal{L}^{(E)} = i \left.\frac{\partial}{\partial t_2}\right|_{t_2=t_1} \mathcal{U}^{(E)}(t_2, t_1) \quad , \tag{5.44}$$

too, can be written as a sum

$$\mathcal{L}^{(E)} = \mathcal{L}^{(E)}_{\text{diag}} + \delta\mathcal{L}^{(E)} \quad . \tag{5.45}$$

It is the latter contribution, $\delta\mathcal{L}^{(E)}$, which is responsible for the mixture of particle and antiparticle states, and hence for pair creation. For weak fields it may be treated as a small perturbation. Using (5.34, 5.35), and defining

$$\dot\beta_{11} := \left.\dot\beta_{21}\right|_{t_2=t_1} \quad , \tag{5.46}$$

one finds that $\delta\mathcal{L}^{(E)}$ is determined by

$$\begin{aligned} \delta\mathcal{L}^{(E)} \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} &= i \begin{pmatrix} 0 & \dot\beta_{11} \\ -\dot\beta_{11}^* & 0 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} + O(\beta) \quad , \\ \delta\mathcal{L}^{(E)} \begin{pmatrix} a_1 \\ b_{-1}^\dagger \end{pmatrix} &= i \begin{pmatrix} 0 & \dot\beta_{11}^* \\ -\dot\beta_{11} & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_{-1}^\dagger \end{pmatrix} + O(\beta) \quad . \end{aligned} \tag{5.47}$$

This implies in particular

$$\delta\mathcal{L}^{(E)} N_-(\vec p(t_1)) = i \sum_m \left[\dot\beta_{11} b_{-1} a_1 \;+\; \text{h.c.}\right] \quad . \tag{5.48}$$

(Terms of order β have been neglected.) Hence whereas the 'diagonal' part $\mathcal{L}^{(E)}_{\text{diag}}$ of the Liouvillian does not mix relevant and irrelevant degrees of freedom, the correction $\delta\mathcal{L}^{(E)}$ obviously does:

$$[\delta\mathcal{L}^{(E)}, \mathcal{P}(t)] \neq 0 \quad . \tag{5.49}$$

(Compare this with (4.27).)

Clearly, $\delta\mathcal{L}^{(E)}$ vanishes in the adiabatic limit. Any deviation from this limit, i.e., $\delta\mathcal{L}^{(E)} \neq 0$, may give rise to corrections to the first and memory terms. They will now be considered separately.

5.3.1 Correction to the First Term

To be more explicit, let us write $\alpha(t_2, t_1)$ instead of α_{21} (and likewise for β). The first term of the Robertson equation is now comprised of two parts containing, respectively, $\mathcal{L}^{(E)}_{\text{diag}}$ and $\delta\mathcal{L}^{(E)}$. The first part is given by

$$\begin{aligned} i\langle\!\langle \text{rel}(t)|\mathcal{L}^{(E)}_{\text{diag}}|N_-(\vec{p})\rangle\!\rangle &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \langle\!\langle \text{rel}(t)|\mathcal{U}^{(E)}_{\text{diag}}(-\tau,0)|N_-(\vec{p})\rangle\!\rangle \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} [|\alpha(-\tau,0)|^2 n_-(\vec{p}-q\vec{E}\tau,t)] \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} n_-(\vec{p}-q\vec{E}\tau,t) \\ &= \dot{n}^{\text{acc}}_-(\vec{p},t) \quad . \end{aligned} \tag{5.50}$$

In the second to last step I used $\alpha(0,0) = 1$, as well as $|\alpha|^2 + |\beta|^2 = 1$ and $\beta(0,0) = 0$. The first part is thus identical with the adiabatic result (4.25), i.e., with the usual acceleration term. For the second part, equation (5.48) immediately yields

$$\begin{aligned} i\langle\!\langle \text{rel}(t)|\delta\mathcal{L}^{(E)}|N_-(\vec{p})\rangle\!\rangle &\propto (\langle ba\rangle_{\text{rel}} + \langle a^\dagger b^\dagger\rangle_{\text{rel}}) \\ &= 0 \quad . \end{aligned} \tag{5.51}$$

Hence there is *no* correction; pair creation does *not* manifest itself in the first term of the Robertson equation.

5.3.2 Correction to the Memory Term

In the absence of pair creation, the memory term of the Robertson equation, calculated to lowest (i.e., second) order in perturbation theory, yields the collision term of the quantum Boltzmann equation. Incorporating pair creation may give rise to a correction. By definition, this correction to the memory term will constitute the source term:

$$\dot{n}^{\text{mem}}_-(\vec{p},t) = \dot{n}^{\text{coll}}_-(\vec{p},t) + \dot{n}^{\text{sou}}_-(\vec{p},t) \quad . \tag{5.52}$$

In our original calculation (in chapter 4), perturbation theory was based on the decomposition

$$\mathcal{L} = \mathcal{L}^{(E)} + \alpha\mathcal{V} \tag{5.53}$$

into one contribution ($\mathcal{L}^{(E)}$) which does not mix relevant and irrelevant degrees of freedom, and a remaining contribution ($\alpha\mathcal{V}$) which does. But in the presence of pair creation, this decomposition no longer works (cf. (5.49)). Rather, one has to start from the decomposition

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(E)}_{\text{diag}} + \underbrace{\delta\mathcal{L}^{(E)} + \alpha\mathcal{V}} \\ &= \mathcal{L}^{(E)}_{\text{diag}} + \tilde{\mathcal{V}} \quad , \end{aligned} \tag{5.54}$$

in which we again separate non-mixing ($\mathcal{L}^{(E)}_{\rm diag}$) from mixing ($\tilde{\mathcal{V}}$) contributions. The general formula (3.30) for second order perturbation theory can be taken over, but with the replacements

$$\begin{aligned} \mathcal{U}^{(0)} &\rightarrow \mathcal{U}^{(E)}_{\rm diag} \\ \alpha\mathcal{V} &\rightarrow \alpha\mathcal{V} + \delta\mathcal{L}^{(E)} \quad . \end{aligned} \tag{5.55}$$

Second order perturbation theory may appear to be a very crude approximation. After all, pair creation is a non-perturbative effect. It is important to note, however, that the non-perturbative nature of pair creation is already encoded in the amplitudes $\dot{\beta}$ and thus in the very definition of $\delta\mathcal{L}^{(E)}$. What we are doing here is just a perturbative treatment of the *many*-particle problem, using the inherently non-perturbative solution of the single-particle problem. On this many-particle level second order perturbation theory can even be shown to be exact, provided the observable under consideration is a particle number ($G_a = N_\pm(\vec{p})$) and there is no scattering ($\mathcal{V} = 0$). For then, in the memory term, $\delta\mathcal{L}^{(E)}$ can only map between products $\{a_1^\dagger a_1, b_{-1}b_{-1}^\dagger\}$ and $\{a_1^\dagger b_{-1}^\dagger, b_{-1}a_1\}$, a_1 and b_{-1} carrying opposite quantum numbers. In particular, the irrelevant components $(a_1^\dagger b_{-1}^\dagger, b_{-1}a_1)$ are always mapped to relevant ones $(a_1^\dagger a_1, b_{-1}b_{-1}^\dagger)$, effectively implying

$$\delta\mathcal{L}^{(E)}_{\rm irr}(t) = \mathcal{Q}(t)\delta\mathcal{L}^{(E)}\mathcal{Q}(t) = 0 \quad . \tag{5.56}$$

Consequently, $\delta\mathcal{L}^{(E)}$ does not contribute to the evolution operator $\mathcal{T}$ which enters in the general equation of motion (3.23). Rather, $\mathcal{T}$ is defined entirely in terms of $\mathcal{L}^{(E)}_{\rm diag}$. Hence, for the same reason as in perturbation theory, $\mathcal{T}$ may effectively be replaced by $\mathcal{U}^{(E)}_{\rm diag}$.

Compared to the ordinary collision term ($\delta\mathcal{L}^{(E)} = 0$) there will be three corrections, corresponding (in (3.30)) to the three combinations

- $\langle\!\langle \mathrm{rel}|\mathcal{V}\ldots\delta\mathcal{L}^{(E)}|N\rangle\!\rangle$,
- $\langle\!\langle \mathrm{rel}|\delta\mathcal{L}^{(E)}\ldots\mathcal{V}|N\rangle\!\rangle$, and
- $\langle\!\langle \mathrm{rel}|\delta\mathcal{L}^{(E)}\ldots\delta\mathcal{L}^{(E)}|N\rangle\!\rangle$.

However, the first two corrections can easily be shown to vanish. They are expectation values of products of field operators, one of which — by virtue of $\delta\mathcal{L}^{(E)}$ — has been converted into an antiparticle operator. The products thus contain an odd number of both particle and antiparticle operators, rendering the expectation value zero. This finding is intuitively obvious for a different reason: the correction which describes pair creation should not depend on the scattering operator $\mathcal{V}$. (Interference terms intertwining pair creation ($\delta\mathcal{L}^{(E)}$) with scattering ($\mathcal{V}$) do exist, though, at higher order in perturbation theory.)

There remains only the third correction. Using $\mathcal{P}\delta\mathcal{L}^{(E)}\mathcal{P} = 0$ and substituting $\tau = t - t'$, it yields the *source term:*

$$\dot{n}_-^{\rm sou}(\vec{p},t) = -\int_0^{t-t_0} {\rm d}\tau\ \langle\!\langle {\rm rel}(t-\tau)|\delta\mathcal{L}^{(E)}\mathcal{U}_{\rm diag}^{(E)}(-\tau,0)\delta\mathcal{L}^{(E)}|N_-(\vec{p})\rangle\!\rangle \tag{5.57}$$

The properties of $\mathcal{U}_{\rm diag}^{(E)}$ and $\delta\mathcal{L}^{(E)}$ then lead to

$$\dot{n}_-^{\rm sou}(\vec{p},t) = 4\,{\rm Re}\int_0^{t-t_0} {\rm d}\tau\,\dot{\beta}^*(-\tau,-\tau)\,[\alpha^*(-\tau,0)]^2{\rm e}^{-i2\phi(-\tau,0)}\,\dot{\beta}(0,0)\,S[\vec{p}](t,\tau) \tag{5.58}$$

with

$$\begin{aligned} S[\vec{p}](t,\tau) &:= 1 - \tfrac{1}{2}n_-(\vec{p}-q\vec{E}\tau,t-\tau) - \tfrac{1}{2}n_+(-\vec{p}+q\vec{E}\tau,t-\tau) \\ &= \left[1 - \tfrac{1}{2}n_-(\vec{p}-q\vec{E}\tau,t-\tau)\right]\left[1 - \tfrac{1}{2}n_+(-\vec{p}+q\vec{E}\tau,t-\tau)\right] - \\ &\quad -\tfrac{1}{4}n_-(\vec{p}-q\vec{E}\tau,t-\tau)n_+(-\vec{p}+q\vec{E}\tau,t-\tau) \quad . \end{aligned} \tag{5.59}$$

The factor S consists of two parts: (i) a product of Pauli blocking factors, governing the *creation* of particles; and (ii) a term proportional to n_-n_+, with negative sign, which is responsible for the *annihilation* of particles. These two contributions cancel when $n_- = n_+ = 1$; i.e., — taking the spin degeneracy into account — when all states are *half* filled. Only then are creation and annihilation in detailed balance. Such a configuration corresponds to infinite temperature, because the occupation numbers in a Fermi distribution, $[1+\exp(\beta\epsilon)]^{-1}$, tend to $1/2$ as $T\to\infty$ ($\beta \to 0$). Therefore, if an electric field could be maintained for an infinite time, it would eventually produce a particle-antiparticle plasma with infinite temperature. In this sense, then, the external field may be regarded as a 'heat pump.' Of course, maintaining a strong field for an infinite time requires an infinite supply of energy. Such an ideal situation might be (approximately) realised in some cosmological systems, but certainly not in heavy ion collisions.

5.4 Consistency with the Schwinger Formula

Before proceeding with the time scale analysis it is important to check that the above source term is consistent with the Schwinger formula (5.1).

Again I assume that the electric field is weak and that hence $|\beta| \approx 0$. As an immediate consequence of this weak field limit, the differential equation (5.29) implies $|\dot{\alpha}_{21}| \approx 0$; which permits us to set

$$\alpha_{21} = 1 \quad . \tag{5.60}$$

Then equation (5.29) further implies

$$\dot{\beta}_{21} = \dot{\beta}_{22}\,{\rm e}^{+i2\phi_{21}} \quad . \tag{5.61}$$

Assuming that (i) $t_0 \to -\infty$ and (ii) the system is dilute ($S = 1$), and using the decomposition

$$\phi(-\tau, 0) = \phi(-\tau, -\infty) - \phi(0, -\infty) \quad , \tag{5.62}$$

the source term reduces to

$$\begin{aligned} \dot{n}_-^{\text{sou}}(\vec{p}) &= 4\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau\, \dot{\beta}^*(-\tau, -\infty)\dot{\beta}(0, -\infty) \\ &= 4\,\mathrm{Re}\left[\beta^*(0, -\infty)\dot{\beta}(0, -\infty)\right] \\ &= 2\, \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} |\beta(\tau, -\infty)|^2 \quad . \end{aligned} \tag{5.63}$$

Of course, the amplitude $\beta(t_2, t_1)$ depends on the momentum $\vec{p}$ through the initial condition $\vec{p}(t = 0) = \vec{p}$. The numerical factor 2 corresponds to the two possible spin orientations.

In order to obtain the *total* production rate per unit volume, we now have to consider

$$\begin{aligned} w &:= \frac{1}{\Omega}\sum_{\vec{p}} \dot{n}_-^{\text{sou}}(\vec{p}) \\ &\to \hbar^{-3}\int \mathrm{d}^3p\; \dot{n}_-^{\text{sou}}(\vec{p}) \\ &= \frac{1}{8\pi^2\hbar^3}\int_{m^2}^\infty \mathrm{d}(\epsilon_\perp^2)\int_{-\infty}^\infty \mathrm{d}p_{||}\, \dot{n}_-^{\text{sou}}(\vec{p}) \quad . \end{aligned} \tag{5.64}$$

Noticing that in the source term,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} |\beta(\tau, -\infty)|^2 &= qE\, \frac{\mathrm{d}}{\mathrm{d}p_{||}(\tau)}\bigg|_{p_{||}(\tau)=p_{||}} \left|\beta[p_{||}(\tau), p_{||}(-\infty)]\right|^2 \\ &= qE \frac{\mathrm{d}}{\mathrm{d}p_{||}} \left|\beta[p_{||}, p_{||}(-\infty)]\right|^2 \quad , \end{aligned} \tag{5.65}$$

the integration over the longitudinal momentum is straightforward:

$$\int_{-\infty}^\infty \mathrm{d}p_{||}\, \dot{n}_-^{\text{sou}}(\vec{p}) = 2qE \cdot |\beta(\infty, -\infty)|^2 \quad . \tag{5.66}$$

The modulus squared, $|\beta|^2$, represents the probability that — for a given transverse momentum $\vec{p}_\perp$ — a non-adiabatic transition will *ever* take place. It is given by the Landau-Zener formula

$$|\beta(\infty, -\infty)|^2 = \exp\left(-\frac{\pi\epsilon_\perp^2}{\hbar qE}\right) \tag{5.67}$$

[103, 104, 105, 106, 107]. The subsequent integration over $(\epsilon_\perp^2)$ is easily performed, yielding the final result

$$w = \frac{(qE)^2}{4\pi^3\hbar^2} \exp\left(-\frac{\pi m^2}{\hbar qE}\right) \quad . \tag{5.68}$$

This result does indeed agree with the leading term ($n = 1$) in Schwinger's formula. As we suspected earlier, the weak field approximation amounts to an omission of the higher order contributions ($n \geq 2$). The weak field approximation is justified if $qE \ll m^2/\hbar$; which for the case of electrons and positrons ($m \approx 500$ keV) means $E < 1$ kV/fm.

5.5 Time Scale Analysis

The source term (5.58), like the collision term, exhibits two characteristic time scales: (i) the *memory time* $\tau_{\text{mem}}(\vec{p})$, which corresponds to the temporal extent of each individual creation process; and (ii) the *production interval* $\tau_{\text{prod}}(\vec{p})$, which corresponds to the average time that elapses *between* successive creation processes. The latter is the inverse of the production rate and as such constitutes the time scale on which the relevant observables — namely the occupation numbers — evolve. Both time scales refer to the specific momentum $\vec{p}$ with which the particles are produced. Hence, strictly speaking, the source term needs to be inspected, and the respective time scales be determined, for each momentum $\vec{p}$. However, from (5.29) we infer that the factor

$$\dot{\beta}(0,0) \propto \frac{\epsilon_\perp}{p_{||}^2 + \epsilon_\perp^2} \tag{5.69}$$

is essentially a Lorentz distribution peaked around $p_{||} = 0$, indicating that most particles are produced with approximately zero longitudinal momentum. It is thus well justified to restrict our attention to the special, yet predominant, case $p_{||} = 0$. (In contrast, the transverse momentum $\vec{p}_\perp$ will not be fixed.)

5.5.1 Memory Time

For $p_{||} = 0$, the factor

$$\dot{\beta}^*(-\tau, -\tau) \propto \frac{(\epsilon_\perp/qE)}{\tau^2 + (\epsilon_\perp/qE)^2} \tag{5.70}$$

constitutes yet another Lorentz distribution, this time in τ, with width $(\epsilon_\perp/qE)$. Only times τ which are smaller than this width can contribute significantly to the source term. One immediately concludes:

$$\tau_{\text{mem}}^{\text{class}}(0, \vec{p}_\perp) = \frac{\epsilon_\perp}{qE} \quad . \tag{5.71}$$

At first sight, this result may seem surprising in that it does not depend on $\hbar$: the memory time is entirely *classical*, despite the fact that the process under consideration — pair creation — is a pure quantum effect. Rather, one might have expected the memory time to be determined via the time-energy uncertainty relation, i.e.,

$$\tau_{\mathrm{mem}}^{\mathrm{quant}}(0,\vec{p}_\perp) \sim \frac{\hbar}{\epsilon_\perp} \quad . \tag{5.72}$$

Physically, this quantum mechanical time scale corresponds to the time needed to create a *virtual* particle-antiparticle pair, which may subsequently reach mass-shell due to acceleration in the external field. It may thus be regarded as the time between two 'tunnelling attempts.' The fact that this time scale does not manifest itself in the source term is a consequence of the weak field approximation; because in this approximation,

$$\frac{\tau_{\mathrm{mem}}^{\mathrm{quant}}}{\tau_{\mathrm{mem}}^{\mathrm{class}}} = \frac{\hbar qE}{\epsilon_\perp^2} < \frac{\hbar qE}{m^2} \ll 1 \quad . \tag{5.73}$$

Hence any contribution of $\tau_{\mathrm{mem}}^{\mathrm{quant}}$ to the exact memory time τ_{mem} is negligible compared to $\tau_{\mathrm{mem}}^{\mathrm{class}}$.

The classical memory time may be interpreted in various ways, depending on the picture employed to visualize the pair creation process:

- In the spin-flip picture, pair creation corresponds to a non-adiabatic transition between two time-dependent energy levels (figure 5.1). Such a transition occurs predominantly in the region of closest approach of the two levels. The width of this transition region corresponds to the memory time.

- Alternatively, pair creation may be viewed as a quantum mechanical tunnelling process (figure 5.2). In this picture, the memory time coincides with the minimum time needed for the wave function to propagate through the barrier (figure 5.3).

5.5.2 Production Interval

Provided the memory time is sufficiently small, one can (i) take the Markovian limit, $S[0,\vec{p}_\perp](t,\tau) \to S[0,\vec{p}_\perp](t,0)$, and (ii) do the quasistationary approximation, $(t-t_0) \to \infty$. The system need *not* be dilute. Again invoking the weak field limit, the same reasoning which led to (5.63) now yields

$$\begin{aligned} \dot{n}_-^{\mathrm{sou}}(0,\vec{p}_\perp,t) &= 4\,\mathrm{Re}\left[\beta^*(0,-\infty)\dot{\beta}(0,-\infty)\right] S[0,\vec{p}_\perp](t,0) \\ &\le 4\,|\beta(0,-\infty)|\cdot|\dot{\beta}(0,-\infty)|\,S[0,\vec{p}_\perp](t,0) \quad . \end{aligned} \tag{5.74}$$

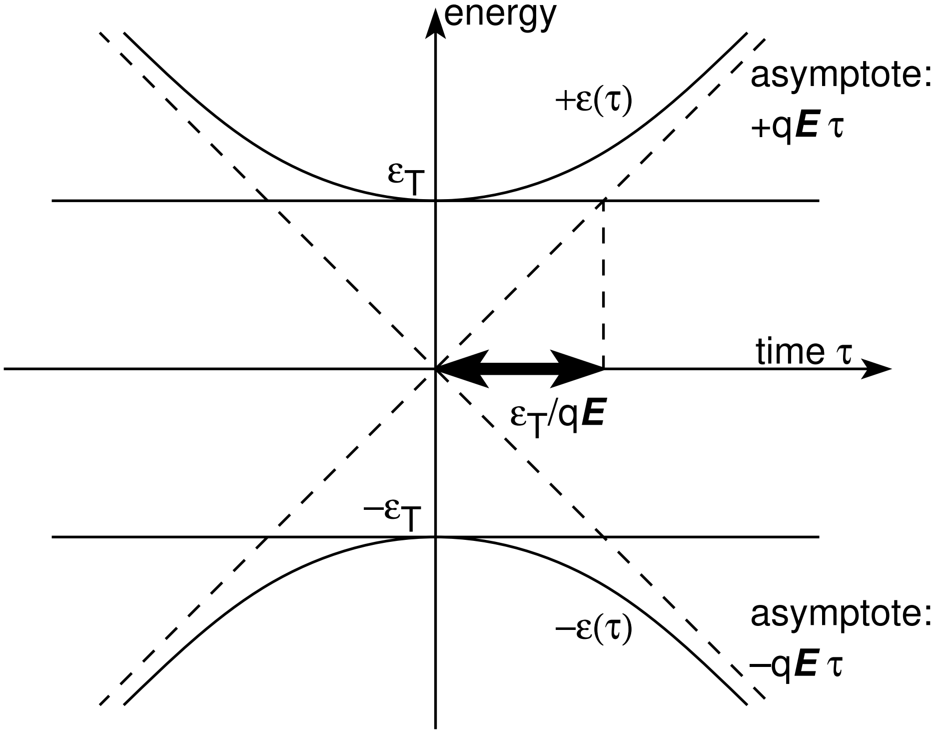

Figure 5.1: Pair creation viewed as a non-adiabatic transition between two time-dependent energy levels. The temporal extent of the transition region is of the order $\epsilon_{\perp}/qE$.

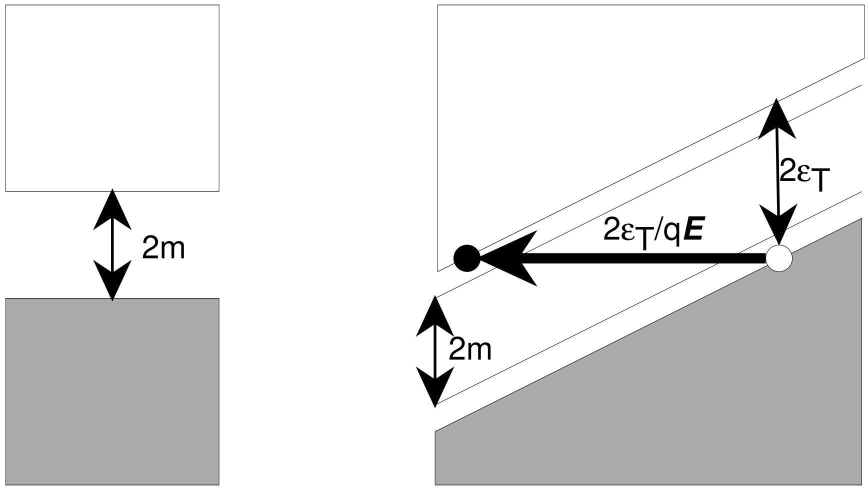

Figure 5.2: The spectrum of the free Dirac-Hamiltonian is comprised of a positive and a negative energy continuum, separated by a gap of magnitude $2m$*. (This gap widens to* $2\epsilon_\perp$ *if one fixes the transverse momentum.) In the ground state the negative energy continuum ('Dirac sea') is completely filled. Applying a homogeneous electric field 'skews' these energy levels, due to the linearly rising potential. It is then possible for an electron to tunnel from the lower to the upper continuum, a process which corresponds to pair creation.*

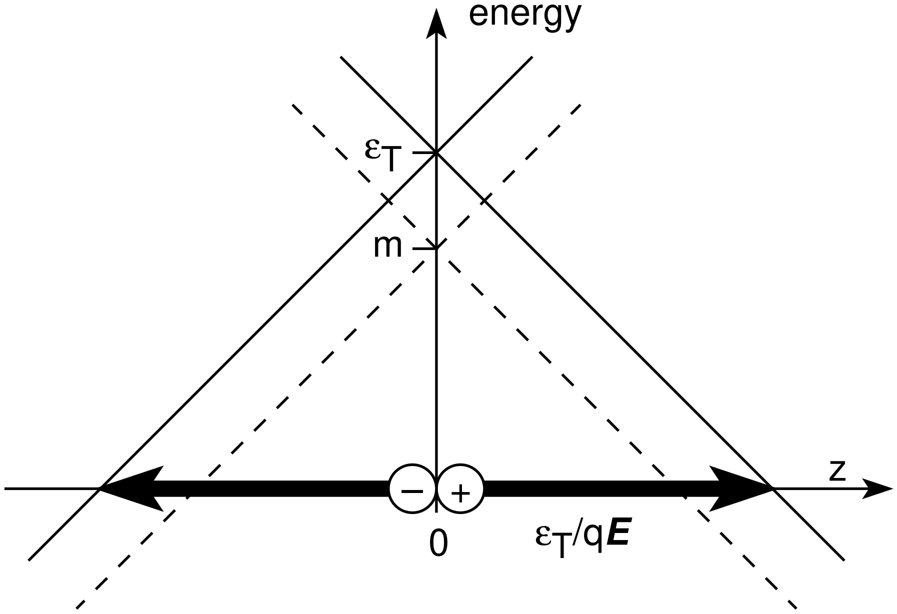

Figure 5.3: In a slightly modified tunnelling picture, a virtual electron-positron pair is produced with zero energy. These virtual particles are subsequently being accelerated by the electric field until they are on mass-shell, i.e., until they have reached the positive energy continuum. The distance each particle has to travel before thus materializing as a 'real' particle is $\epsilon_\perp/qE$. Given that their respective wave functions can propagate at most with the speed of light ($c = 1$), $\epsilon_\perp/qE$ is also the minimum time needed for the wave function to penetrate through the barrier.

Since $p_{\|} = 0$, a non-adiabatic transition during the time interval $[-\infty, 0]$ is as likely as a transition during $[0, \infty]$, and hence

$$|\beta(0,-\infty)|^2 = \tfrac{1}{2}|\beta(\infty,-\infty)|^2 \quad . \tag{5.75}$$

Therefore the factor $|\beta|$ which appears in the above source term can be related to the Landau-Zener formula; whereas the factor $|\dot{\beta}|$ is determined by the differential equation (5.29) (with $\alpha = 1$). One finds

$$\dot{n}_-^{\mathrm{sou}}(0,\vec{p}_\perp,t) \leq \sqrt{2}\,\frac{qE}{\epsilon_\perp}\exp\left(-\frac{\pi\epsilon_\perp^2}{2\hbar qE}\right)\, S[0,\vec{p}_\perp](t,0) \quad . \tag{5.76}$$

Inversion immediately yields a lower bound for the production interval:

$$\tau_{\mathrm{prod}}(0,\vec{p}_\perp) \geq \frac{1}{\sqrt{2}S[0,\vec{p}_\perp]}\,\frac{\epsilon_\perp}{qE}\exp\left(\frac{\pi\epsilon_\perp^2}{2\hbar qE}\right) \tag{5.77}$$

The production interval thus depends not only on the strength of the external field but also — through S — on the number of particles already produced. As is intuitively obvious for fermions, particle creation becomes less frequent (i.e., τ_{prod} grows) as more states are filled (i.e., S decreases).

5.6 Resumé: Is Particle Creation Irreversible?

Both the memory time and the production interval decrease with increasing field strength. But the form of their E-dependence differs: $(1/E)$ vs. $(1/E)\exp(c/E)$; their ratio is of the order

$$\frac{\tau_{\mathrm{mem}}^{\mathrm{class}}}{\tau_{\mathrm{prod}}}(0,\vec{p}_\perp) \sim S[0,\vec{p}_\perp]\,\exp\left(-\frac{\pi\epsilon_\perp^2}{2\hbar qE}\right) \quad . \tag{5.78}$$

In the weak field limit, i.e., when $\hbar qE \ll m^2 \leq \epsilon_\perp^2$, the particle creation process is thus Markovian: $\tau_{\mathrm{mem}}/\tau_{\mathrm{prod}} \ll 1$. The Markovian property, in turn, implies an H-theorem and hence is responsible for the perceived irreversibility of particle creation.

Of course, as we discussed in detail in chapter 2, this apparent irreversibility is a consequence of choosing a reduced level of description: information is being transferred from accessible (slow) to inaccessible (fast) degrees of freedom. In the case of particle creation, the slow degrees of freedom (i.e., the observables spanning the Markovian level of description) are the occupation numbers of the various momentum states. From these, information gradually 'leaks' into unobserved degrees of freedom: correlations and rapidly oscillating phases which entangle the respective wave functions of the members of a particle-antiparticle pair.

Chapter 6

Conclusions

In this chapter, I will summarize important results (section 6.1), outline future tasks (section 6.2), and discuss the findings of this study in the context of my original agenda (section 6.3).

6.1 Main Results

The main results of this study may be grouped into three categories: (i) the complete set of time scales governing the evolution of the system, and thus a precise characterisation of the Markovian level of description; (ii) the associated Markovian entropy; and (iii) the associated kinetic equation, including a source term derived from first principles. These results will be reviewed separately in the following three subsections.

Time Scales

In my model, there is no time scale associated with acceleration. In contrast, there are three characteristic time scales associated with collisions (cf. section 4.3):

- the average time needed for a particle to pass through an interaction range,
$$\tau_{\mathrm{range}} := \frac{\eta}{\sqrt{\langle v^2 \rangle}} \quad , \tag{6.1}$$
where η denotes the range of the interaction and v the particle velocity;
- the quantum mechanical 'off-shell' time
$$\tau_{\mathrm{off}} := \hbar\beta \quad , \tag{6.2}$$
where β denotes the inverse temperature; and
- the average time which elapses between two successive collisions (and which is thus proportional to the mean free path)
$$\tau_{\mathrm{free}} := \frac{1}{n\langle v\sigma \rangle} \quad , \tag{6.3}$$
where n denotes the particle density and σ the total scattering cross section.

Likewise, there are three characteristic time scales associated with particle creation (cf. section 5.5):

- the time between successive tunnelling attempts,

$$\tau_{\text{attempt}} := \frac{\hbar}{\epsilon_\perp} \quad , \tag{6.4}$$

where $\epsilon_\perp$ denotes the transverse energy of the (virtual) particles;

- the tunnelling time proper,

$$\tau_{\text{tunnel}} := \frac{\epsilon_\perp}{qE} \quad , \tag{6.5}$$

where q denotes the charge and E the strength of the electric field; and

- the production interval

$$\tau_{\text{prod}} \sim \frac{1}{S[0,\vec{p}_\perp]}\,\frac{\epsilon_\perp}{qE}\exp\left(\frac{\pi\epsilon_\perp^2}{2\hbar qE}\right) \quad , \tag{6.6}$$

where the dimensionless factor S is given by (5.59). (There may be an additional numerical pre-factor of order unity, which has not been determined.)

On the level of description spanned by the number operators $\{N_\pm(\vec{p})\}$, the times τ_{range}, τ_{off}, τ_{attempt} and τ_{tunnel} represent contributions to the total memory time; whereas τ_{free} and τ_{prod} constitute the time scales on which the expectation values of the relevant observables (i.e., the occupation numbers) evolve. Therefore, the level of description is Markovian if and only if

$$\tau_{\text{range}}\,,\ \tau_{\text{off}}\,,\ \tau_{\text{attempt}}\,,\ \tau_{\text{tunnel}} \ll \tau_{\text{free}}\,,\ \tau_{\text{prod}} \quad . \tag{6.7}$$

Qualitatively, the validity of the Markovian property is guaranteed if the system is sufficiently dilute or weakly interacting (i.e., $n\sigma$ is small), and the electric field is sufficiently weak. Precise quantitative conditions can be obtained by inserting the formulas for the various time scales into the above inequality. Provided these conditions are satisfied, the level of description spanned by the number operators $\{N_\pm(\vec{p})\}$ becomes the Markovian level of description. It is then possible to obtain a Markovian kinetic equation for the occupation numbers (i.e., the momentum distribution) $n_\pm(\vec{p},t)$.

The relative magnitude of the two slow time scales, τ_{free} and τ_{prod}, depends on both the strength of the electric field and the density of the particles already produced: $\tau_{\text{prod}}/\tau_{\text{free}}$ increases as either E decreases or n increases. There are two special cases:

- If $\tau_{\text{prod}} \ll \tau_{\text{free}}$ then the (relevant) entropy of the system initially grows due to pair creation alone; but this does not lead to complete thermalization, since the particles are not produced with a thermal distribution. (In the weak field limit, the momenta of the produced particles are narrowly peaked around $\vec{p} = 0$ and thus nonrelativistic. Therefore, if their distribution were thermal, it should be Gaussian. This is indeed the case for the transverse components, but *not* for the longitudinal components.) Only on a much longer time scale, namely τ_{free}, is the equilibration completed.

- If $\tau_{\text{prod}} \gg \tau_{\text{free}}$ then the particles may be regarded as being thermalized, due to collisions, almost immediately after their creation. Hence on the (slower) production time scale, it is sufficient to describe the system by very few expectation values such as the total energy, total particle number, and total current. Associated with these expectation values is a smaller level of description. Like the original level of description (spanned by *all* number operators), it is Markovian; but it is applicable only to the evolution of the system over times much longer than τ_{free}.

Markovian Entropy

Associated with the Markovian level of description, spanned by $\{N_\pm(\vec{p})\}$, is the Markovian entropy

$$\begin{aligned} S_m(t) &= -2k\sum_{\vec{p}}\left[\frac{n_-(\vec{p},t)}{2}\ln\frac{n_-(\vec{p},t)}{2} + \left(1-\frac{n_-(\vec{p},t)}{2}\right)\ln\left(1-\frac{n_-(\vec{p},t)}{2}\right)\right] + \\ &\quad + (n_- \leftrightarrow n_+) \end{aligned} \tag{6.8}$$

(cf. (C.23)). The numerical factor 2 reflects the spin degeneracy of electrons and positrons. If the system is initially in the vacuum state, then charge symmetry dictates that at all times the positrons (n_+) and the electrons (n_-) contribute equally to the Markovian entropy. As I discussed in detail in chapter 2, it is this Markovian entropy which obeys an H-theorem.

Under Lorentz transformations, the particle momenta change, $\vec{p} \to \vec{p}\,'$, and so do the occupation numbers: $n'(\vec{p}\,') = n(\vec{p})$. The Markovian level of description, however, remains unchanged:

$$\operatorname{span}\bigcup_{\vec{p}\,'}\{N_\pm(\vec{p}\,')\} = \operatorname{span}\bigcup_{\vec{p}}\{N_\pm(\vec{p})\} \quad . \tag{6.9}$$

This invariance has two important consequences: (i) the Markovian property of the level of description spanned by $\{N_\pm(\vec{p})\}$ is not restricted to one particular frame of reference, despite the fact that the various time scales discussed above are certainly not Lorentz invariant; (ii) the total amount of information carried by all occupation numbers remains unchanged: the Markovian entropy is Lorentz invariant. (This invariance property of the Markovian entropy also follows from the fact that $n_\pm(\vec{p},t)$ are scalar distributions in momentum space, and the summation $\sum_{\vec{p}}$ or integration $\Omega\hbar^{-3}\int \mathrm{d}^3p$, respectively, is Lorentz invariant.)

Since the idealised system considered in this study is assumed to have infinite size, it is more appropriate to consider the Markovian entropy *density*. The entropy density, of course, is no longer Lorentz invariant.

Kinetic Equation

The kinetic equation for the occupation numbers $n_\pm(\vec{p},t)$ is a generalisation of the quantum Boltzmann equation. It contains an additional source term, and may also account for finite memory effects. (These memory effects become negligible in the Markovian limit.) It has the following structure:

$$\left[\frac{\partial}{\partial t} + e\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\vec{E}\cdot\nabla_{\vec{p}}\right]\begin{pmatrix} n_+(\vec{p},t) \\ n_-(\vec{p},t) \end{pmatrix} = \begin{pmatrix} \dot{n}_+^{\mathrm{coll}}(\vec{p},t) \\ \dot{n}_-^{\mathrm{coll}}(\vec{p},t) \end{pmatrix} + \begin{pmatrix} \dot{n}_+^{\mathrm{sou}}(\vec{p},t) \\ \dot{n}_-^{\mathrm{sou}}(\vec{p},t) \end{pmatrix} \tag{6.10}$$

(cf. (4.85)). The acceleration term, which is incorporated into the left-hand side of the kinetic equation, originates from the first term of the Robertson equation. In contrast, both the collision and the source term, and hence the right-hand side of the kinetic equation, have their origin in the memory term of the Robertson equation. Neither the derivatives with respect to time, nor the electric field, are Lorentz covariant. The above kinetic equation is therefore *not* covariant.

The collision term was calculated in chapter 4. There I considered the scattering of identical particles, such as e^-e^- or e^+e^+ scattering. I obtained the general expression (4.44), which may account for finite memory effects; only in the Markovian limit does it reduce to the well-known collision term of the Boltzmann equation. For scattering of distinguishable particles (such as e^+e^- scattering), the calculation proceeds along similar lines, but yields an extra factor of 2 (cf. (4.85)). Thus the complete collision term for, say, electrons reads:

$$\dot{n}_-^{\mathrm{coll}}(\vec{p},t) = 2\frac{\alpha^2}{\hbar^2}\mathrm{Re}\int_0^{t-t_0}\mathrm{d}\tau \sum_{\vec{p}_1\vec{p}_2\vec{p}_3}\left[\tfrac{1}{2}f_{--}(\tau)\cdot C_{--}(t,\tau) + f_{-+}(\tau)\cdot C_{-+}(t,\tau)\right] \tag{6.11}$$

The factors f and C are given by (4.47) and (4.38), respectively. In the Markovian limit, this reduces to (4.85):

$$\dot{n}_-^{\mathrm{coll}}(\vec{p},t) = \sum_{\vec{p}_1\vec{p}_2\vec{p}_3}\left[\tfrac{1}{2}r_{--}\cdot C_{--}(t,0) + r_{-+}\cdot C_{-+}(t,0)\right] \tag{6.12}$$

where the respective transition rates are given by (4.68). The collision term for positrons follows immediately by interchanging subscripts $(+ \leftrightarrow -)$.

The source term was calculated in chapter 5. For electrons, I obtained the expression (5.58):

$$\dot{n}_{-}^{\mathrm{sou}}(\vec{p},t) = 4\,\mathrm{Re}\int_0^{t-t_0} \mathrm{d}\tau\,\dot{\beta}^*(-\tau,-\tau)\,[\alpha^*(-\tau,0)]^2 \mathrm{e}^{-i2\phi(-\tau,0)}\,\dot{\beta}(0,0)\,S[\vec{p}](t,\tau) \tag{6.13}$$

with the factor S given by (5.59), and the coefficients α, β determined by the differential equation (5.29). Due to momentum and charge conservation, the source term for positrons is easily obtained via

$$\dot{n}_{+}^{\mathrm{sou}}(\vec{p},t) = \dot{n}_{-}^{\mathrm{sou}}(-\vec{p},t) \quad . \tag{6.14}$$

According to the Landau-Zener formula (5.67), the transverse momenta of the produced particles are peaked around $\vec{p}_\perp = 0$ with a Gaussian distribution of width $\sqrt{\hbar q E}$. Likewise, the longitudinal momenta are peaked around $p_{||} = 0$, but neither with a Gaussian distribution nor — as hypothesized in [25, 26] — with a δ-function distribution. There is no analytic expression for the form of the $p_{||}$-distribution; but a lower bound for its width can be derived from the Heisenberg uncertainty relation:

$$\Delta p_{||} \geq \frac{\hbar}{l_{\mathrm{tunnel}}} = \frac{\hbar}{\tau_{\mathrm{tunnel}}} \sim \frac{\hbar q E}{m} \tag{6.15}$$

where l_{tunnel} is the 'tunnelling length' (cf. figure 5.3), and $c = 1$.

6.2 Outlook

In this thesis, I presented a rigorous method to determine the Markovian level of description of a system experiencing collisions and pair creation. I proved that the associated relevant entropy must increase, and was thus able to explain the origin of the apparent irreversibility of particle creation. In order to describe the thermalisation of the system *quantitatively*, however, one has to solve the complete kinetic equation with its acceleration, collision and source terms. Because of the axial symmetry of the external potential, the occupation numbers depend only on two independent variables, $p_{||}$ and $\epsilon_\perp$. The kinetic equation thus represents a system of coupled integro-differential equations in two variables. Solving these equations is a challenging task that must be tackled numerically.

In addition, there are various ways in which the QED model that I investigated in this thesis can (and should) be extended. Potential modifications mainly affect three areas: (i) the description of the pair creation process proper; (ii) the nature of, and the interaction of the particles with, the external field; and (iii) the interaction between the particles.

Regarding the pair creation process proper,

- one may go beyond the weak field approximation and include the $O(\beta)$-correction to the off-diagonal part $\delta\mathcal{L}^{(E)}$ of the Liouvillian (cf. (5.47)). This will likely lead to corrections in both the time scales and the source term. In addition, it should enable us to recover the higher order terms in the Schwinger formula (cf. section 5.4).

- Since the Markovian limit is only justified in the weak field approximation, relaxing the weak field approximation makes it necessary to include non-Markovian corrections. It is then desirable to conduct a detailed study of the resulting memory effects.

- In realistic configurations, particle creation occurs in a finite region in space and within a finite time. One may thus wish to incorporate finite-time and finite-size effects [108, 109]. Moreover, the spatial region may vary with time (as is the case for, e.g., the region between two receding nuclei in a heavy ion collision) [110].

Regarding the external field,

- one may want to allow for an explicit time-dependence of the external field. This is necessary in order to model, e.g., particle creation in the expanding early universe [9] or during inflation [111].

- Following their production, particles are accelerated by the external field and thus give rise to a nonvanishing current $\vec{j}$, which in turn generates a field. In the electromagnetic case, the induced field is determined by Maxwell's equations [112]

$$\begin{aligned} \mathrm{div}\vec{E}_{\mathrm{ind}} = 0 \quad &, \quad \mathrm{curl}\vec{B}_{\mathrm{ind}} - \frac{\partial \vec{E}_{\mathrm{ind}}}{\partial t} = 4\pi\vec{j} \\ \mathrm{div}\vec{B}_{\mathrm{ind}} = 0 \quad &, \quad \mathrm{curl}\vec{E}_{\mathrm{ind}} + \frac{\partial \vec{B}_{\mathrm{ind}}}{\partial t} = 0 \quad . \end{aligned} \tag{6.16}$$

  (Due to charge conservation and spatial homogeneity, the charge density ρ vanishes.) The total field is then the sum of the external and this induced field. This 'backreaction' of the produced particles on the electromagnetic field was discussed in [113] and studied numerically by Kluger and Eisenberg [25, 26]. Backreaction gives rise to a collective motion of the produced particles reminiscent of plasma oscillations. Associated with these oscillations is an additional time scale, whose influence on the thermalization properties of the system needs to be studied.

- The quantum nature of the electromagnetic field should be accounted for by incorporating processes involving the emission, absorption, or scattering of real photons. Mathematically, this requires an extension of the Fock space and, consequently, the inclusion of additional Feynman diagrams corresponding to, e.g., bremsstrahlung or annihilation into real photons. Such an extension will yield additional time scales. Furthermore, there will be a (relevant) entropy associated with the photons, which has to be added to the entropy of the charged particles.

- It is desirable to generalise the results of the present study to arbitrary gauge fields, especially QCD, and gravity.

Regarding the interaction between the particles,

- it is more appropriate to describe two-particle interactions in a plasma with a screened Coulomb, rather than a Gaussian, scattering potential. The screening length is generally density dependent and thus a function of time. This will make the matrix elements, and hence the transition rates, complicated (and implicitly time-dependent) functionals of the momentum space distribution. It may lead to modifications of the collision time scales.

- In this study, the time scale analysis of the collision term was performed in the dilute gas approximation, in which case the equilibrium distribution could be approximated by a Maxwell-Boltzmann distribution (cf. section 4.3). One may wish to do the time scale analysis without this approximation, thus accounting for quantum statistics effects.

- In highly relativistic processes, it is not justified to neglect retardation effects; the effective 4-Fermi interaction then has to be replaced by a mediated interaction.

6.3 Closing the Circle

At the end of chapter 2, I laid out a general agenda for the investigation of thermalization properties of an arbitrary physical system. In particular, I emphasized the intimate connection between irreversibility and the existence of strongly separated time scales. In subsequent chapters, that agenda was executed for the special case of a quantum many-particle system in the presence of a homogeneous electric field. The system was assumed to experience acceleration, binary collisions, and — most importantly — pair creation. Careful examination of the time scales then enabled me to give a rigorous explanation for the apparent irreversibility of particle creation. I determined the conditions under which the coarse-grained evolution of the system becomes Markovian, a property which immediately leads to an H-theorem. Moreover, I was able to derive a kinetic equation that, for the first time, includes a source term derived from first principles.

Here I would like to review the original agenda devised in chapter 2, and annotate each step with a brief discussion of the results I obtained for my specific model.

- *Identify all time scales.* In my model, there is no time scale associated with acceleration. In contrast, there are three characteristic time scales associated with collisions: the average time needed for a particle to pass through an interaction range; the quantum mechanical 'off-shell' time; and the average time which elapses between two successive collisions. Likewise, there are three characteristic time scales associated with particle creation: the time between successive tunnelling attempts; the tunnelling time proper; and the production interval.

- *Find the Markovian level of description.* On the level of description spanned by the number operators $\{N_\pm(\vec{p})\}$, the production interval and the average time between collisions constitute the time scales on which the expectation values of the relevant observables (i.e., the occupation numbers) evolve. All the other time scales refer to memory effects, and therefore represent contributions to the memory time. If and only if this memory time is small compared to the time scales on which the occupation numbers evolve, then the level of description becomes Markovian. Precise mathematical conditions for the validity of the Markovian limit were derived in chapters 4 and 5; qualitatively, the many-particle system must be sufficiently dilute or weakly interacting, and the electric field sufficiently weak, for the Markovian property to hold.

- *Find the equilibrium level of description.* Symmetries of the dynamics, such as gauge invariance or axial symmetry, determine the constants of the motion, which in turn span the equilibrium level of description. However, this level is not of practical importance here, because no realistic physical system will ever fully equilibrate due to particle creation alone.

- *Obtain the rate of increase of the Markovian entropy.* Associated with the reduced level of description, spanned by the number operators, is a relevant entropy. I showed that in the Markovian limit, this entropy increases (H-theorem). I was thus able to explain the origin of the apparent irreversibility of particle creation. Quantitative predictions for the magnitude of the entropy growth rate, however, will require numerical calculations. I provided the basis for such a numerical study by deriving, from first principles, a complete kinetic equation that incorporates not only acceleration and collisions, but also particle creation.

- *Obtain the equilibration time, provided the thermalization is complete.* The system will not fully thermalize due to particle creation alone. It is therefore necessary to study the interplay of particle creation with collisions, a task that has to be tackled numerically. Again, such a study may be performed on the basis of the kinetic equation which was derived in this thesis.

Appendix A

Plane Wave Normalisations and Continuum Limit

Single-particle wave functions $|\psi\rangle$ are often represented on either position space or momentum space, the respective bases in Hilbert space being $\{|\vec{x}\,\rangle\}$ or $\{|\vec{p}\,\rangle\}$. For simplicity, spin or other discrete quantum numbers are omitted. Clearly, both bases must span the entire Hilbert space and hence must satisfy the completeness property

$$\int_\Omega |\vec{x}\,\rangle\langle\vec{x}\,| = \int_{\vec{p}} |\vec{p}\,\rangle\langle\vec{p}\,| = 1 \quad . \tag{A.1}$$

Here $\int_\Omega$ denotes the integration over the spatial volume Ω, and $\int_{\vec{p}}$ the summation or integration over all momenta.

In position space,

$$\int_\Omega = \int \mathrm{d}^3x \quad , \tag{A.2}$$

$$\langle\vec{x}\,|\vec{y}\,\rangle = \delta^3(\vec{x}-\vec{y}) \quad . \tag{A.3}$$

In momentum space, one has to distinguish two cases: (i) the discrete case (Ω finite), and (ii) the continuous case ($\Omega\to\infty$). In the *discrete* case,

$$\int_{\vec{p}} = \sum_{\vec{p}} \quad , \tag{A.4}$$

$$\langle\vec{p}\,|\vec{k}\,\rangle = \delta_{\vec{p}\vec{k}} \quad , \tag{A.5}$$

$$\langle\vec{x}\,|\vec{p}\,\rangle = \frac{1}{\sqrt{\Omega}}\exp\left[\frac{i}{\hbar}\vec{p}\cdot\vec{x}\right] \quad . \tag{A.6}$$

In the *continuous* case, the momentum space integration is defined only up to an arbitrary normalisation factor $N(\vec{p})$:

$$\int_{\vec{p}} = \hbar^{-3}\int \mathrm{d}^3p\; N(\vec{p}) \quad , \tag{A.7}$$

$$\langle\vec{p}\,|\vec{k}\,\rangle = \frac{\hbar^3}{N(\vec{p})}\delta^3(\vec{p}-\vec{k}) \quad , \tag{A.8}$$

$$\langle\vec{x}\,|\vec{p}\,\rangle = \frac{1}{\sqrt{N(\vec{p})}}\exp\left[\frac{i}{\hbar}\vec{p}\cdot\vec{x}\right] \quad . \tag{A.9}$$

Here $đ^n p$ and $\bar{\delta}^n(\vec{p})$ are shorthand notations for $d^n p/(2\pi)^n$ and $(2\pi)^n \delta^n(\vec{p})$, respectively [86]. Some common choices for the normalisation factor are $N(\vec{p}) = 1$, $(2\pi\hbar)^3$, Ω, $1/(2E)$. The latter two choices render $\int_{\vec{p}}$ Lorentz-invariant.

Independent of the choice of the normalisation factor, one finds that the *number of states* is given by

$$\begin{aligned} \mathrm{tr}\left(\int_{\vec{p}} |\vec{p}\,\rangle\langle\vec{p}\,|\right) &= \int_\Omega \int_{\vec{p}} \langle\vec{x}\,|\vec{p}\,\rangle\langle\vec{p}\,|\vec{x}\,\rangle \\ &= \begin{cases} \sum_{\vec{p}} & : \text{ discrete} \\ \Omega\hbar^{-3}\int đ^3 p & : \text{ continuous} \end{cases} \quad . \end{aligned} \tag{A.10}$$

Hence when taking the continuum limit, the appropriate substitutions are

$$\sum_{\vec{p}} \to \frac{\Omega}{\hbar^3}\int đ^3 p \quad \text{and} \quad \delta_{\vec{p}\vec{k}} \to \frac{\hbar^3}{\Omega}\bar{\delta}^3(\vec{p}-\vec{k}) \tag{A.11}$$

(which correspond to $N(\vec{p}) = \Omega$).

Appendix B

A Brief Review of Quantum Statistical Mechanics

B.1 Statistical Mechanics = Applied Statistical Inference

Typically, all the information available about a macroscopic system is a (relatively small) set of expectation values

$$g_a := \langle G_a \rangle \quad . \tag{B.1}$$

From this partial information one wants to infer other properties of the system. This is the task typically encountered in statistical inference; it is an entirely statistical problem and has nothing to do with physics. The given expectation values put constraints on the statistical operator, but they do not specify it uniquely. So which among all the allowed statistical operators are we to choose?

The answer is provided by the *Maximum Entropy Principle*. We want to choose a statistical operator that satisfies the constraints but beyond that carries no additional information. In other words, we want to make the 'least biased choice', putting into the statistical operator no more information than what is furnished by the expectation values $\{g_a\}$. Mathematically this is achieved by maximizing the entropy $S[\rho]$ under the constraints $\mathrm{tr}\,\rho = 1$ and $\mathrm{tr}\,(\rho G_a) = g_a$. Using the method of Lagrange multipliers (and absorbing a factor $-k$ into their definition) we obtain

$$\delta\left[\mathrm{tr}\,(\rho \ln \rho) + \eta \mathrm{tr}\,\rho + \lambda^a \mathrm{tr}\,(\rho G_a)\right] = 0 \tag{B.2}$$

which leads to the generalised canonical statistical operator (or Boltzmann-Gibbs distribution)

$$\rho = Z^{-1} \exp(-\lambda^a G_a) \quad . \tag{B.3}$$

To simplify notation, we have employed Einstein's summation convention. Z is the partition function

$$Z = \mathrm{tr}\, \exp(-\lambda^a G_a) \quad . \tag{B.4}$$

Operators $G_a[h^b]$ may depend on external parameters h^b such as volume, external electromagnetic fields etc. We define the conjugate parameters

$$\gamma_b := -\lambda^a \langle \frac{\partial G_a}{\partial h^b} \rangle \quad . \tag{B.5}$$

Both the expectation values g_a and the parameters γ_a can be obtained directly from the partition function:

$$g_a = -\left(\frac{\partial}{\partial \lambda^a} \ln Z \right)_{\lambda,h} \tag{B.6}$$

$$\gamma_a = \left(\frac{\partial}{\partial h^a} \ln Z \right)_{\lambda,h} \tag{B.7}$$

The subscripts λ, h indicate that when taking the derivative, all the remaining parameters must be kept fixed. Notice the different signs in the two formulae. Inserting the generalised canonical statistical operator into the definition for the entropy we obtain

$$S = k \ln Z + k\lambda^a g_a \quad , \tag{B.8}$$

$$\mathrm{d}S = k\lambda^a \mathrm{d}g_a + k\gamma_b \mathrm{d}h^b \quad . \tag{B.9}$$

In the particular case of a *homogeneous system* the external parameters h^b are extensive, and the entropy scales with g_a, h^b:

$$S_{\mathrm{hom}}(\tau g_a, \tau h^b) = \tau \cdot S_{\mathrm{hom}}(g_a, h^b) \quad \forall \quad \tau > 0 \quad . \tag{B.10}$$

This leads to

$$S_{\mathrm{hom}} = k\lambda^a g_a + k\gamma_b h^b \tag{B.11}$$

and the Gibbs-Duhem relation

$$g_a \mathrm{d}\lambda^a + h^b \mathrm{d}\gamma_b = 0 \quad . \tag{B.12}$$

B.2 Energy and Temperature

An observable of particular importance is the *energy* of the system. (Its importance stems from the fact that it is a constant of the motion.) It corresponds to the pair

$$(g, \lambda) \to (U, \beta = 1/kT) \quad . \tag{B.13}$$

For the remaining observables g_a, h^b we define new parameters

$$l^a := \lambda^a/\beta \quad , \quad m_a := \gamma_a/\beta \quad ; \tag{B.14}$$

some important examples are listed in table B.1. Then

g, h	λ, γ	l, m	Names
U	β		energy, (inverse) temperature
V		p	volume, pressure
N	$(-\alpha)$	$-\mu$	particle number, chemical potential
$\vec{E}$		$\vec{P}$	electric field, electric polarization
$\vec{P}$		$-\vec{E}$	
$\vec{B}$		$\vec{M}$	magnetic induction, magnetization
$\vec{M}$		$-\vec{B}$	
$\vec{p}$		$-\vec{v}$	momentum, velocity
π^μ		u_μ	kinetic 4-momentum, 4-velocity
$\vec{L}$		$-\vec{\omega}$	angular momentum, angular velocity

Table B.1: Some typical thermodynamic variables. The kinetic four-momentum is given by $\pi^\mu = p^\mu + q/c \cdot A^\mu$.

$$\begin{aligned} \mathrm{d}U &= \underbrace{T\mathrm{d}S}\ \underbrace{-l^a \mathrm{d}g_a - m_b \mathrm{d}h^b} \\ &= \delta Q + \delta A \quad . \end{aligned} \tag{B.15}$$

We define a generalised thermodynamic potential

$$\Phi := U - TS + l^a g_a \quad , \tag{B.16}$$

$$\mathrm{d}\Phi = -S\mathrm{d}T + g_a \mathrm{d}l^a - m_b \mathrm{d}h^b \quad . \tag{B.17}$$

This last relation shows that S, g_a and m_b can be obtained from the potential by differentiation, provided Φ is given in terms of the temperature T, l^a and h^b. In fact, due to (B.8),

$$\Phi = -\frac{1}{\beta} \ln Z \quad . \tag{B.18}$$

This permits to write the statistical operator as

$$\rho = \exp[\beta(\Phi - H - l^a G_a)] \quad . \tag{B.19}$$

For a *homogeneous system*

$$\Phi_{\mathrm{hom}} = -m_b h^b \quad , \tag{B.20}$$

and the Gibbs-Duhem relation reads

$$S\mathrm{d}T - g_a \mathrm{d}l^a - h^b \mathrm{d}m_b = 0 \quad . \tag{B.21}$$

B.3 Correlations

In quantum statistics observables need not commute, and therefore the usual correlation function $\langle AB \rangle$ is in general not symmetric. One may, however, define a *symmetrized correlation*

$$\langle A;B\rangle := \int_0^1 \mathrm{d}\mu \,\mathrm{tr}\left[\frac{\exp(-\lambda\cdot G)}{Z}\exp(\mu\lambda\cdot G)A^\dagger\exp(-\mu\lambda\cdot G)B\right] \quad , \tag{B.22}$$

where $\lambda \cdot G$ stands for $\lambda^a G_a$. This symmetrized correlation is symmetric for Hermitian observables and positive definite; in fact, it has all properties of a scalar product. In case either A or B commutes with $\lambda^a G_a$, it reduces to the usual correlation $\langle AB \rangle$. Some further useful properties are

$$\langle 1;A\rangle = \langle A\rangle \quad , \tag{B.23}$$

$$\langle A;G_a\rangle = g_a\langle A\rangle^* - \frac{\partial\langle A\rangle^*}{\partial\lambda^a} \quad , \tag{B.24}$$

for arbitrary operators A. Denoting

$$\delta A := A - \langle A\rangle \tag{B.25}$$

one defines a *correlation matrix*

$$C_{ab} := \langle \delta G_a;\delta G_b\rangle \quad . \tag{B.26}$$

The correlation matrix is Hermitian and has non-negative eigenvalues:

$$C^\dagger = C \quad , \quad C\geq 0 \quad . \tag{B.27}$$

An arbitrary matrix element is given by

$$C_{ab} = -\left(\frac{\partial g_a^*}{\partial\lambda^b}\right)_{\lambda,h} \quad . \tag{B.28}$$

Positivity of the diagonal elements implies that

$$\left(\frac{\partial g_{(a)}^*}{\partial\lambda^{(a)}}\right)_{\lambda,h} \leq 0 \quad . \tag{B.29}$$

(We put the index a in brackets to indicate that it is not to be summed over.) Provided C is invertible, it can be shown that

$$\left(\frac{\partial g_a^*}{\partial\lambda^b}\right)_{g,h} = -\frac{1}{(C^{-1})^{ba}} \tag{B.30}$$

where now the remaining $\{g\}$ rather than $\{\lambda\}$ are kept fixed. Since $C \geq 0$, its inverse, too, must have non-negative eigenvalues, and in particular the diagonal elements of C^{-1} must be non-negative; hence we also have

$$\left(\frac{\partial g^*_{(a)}}{\partial \lambda^{(a)}}\right)_{g,h} \leq 0 \quad . \tag{B.31}$$

In case the energy is one of the observables considered, our last result implies that the heat capacity

$$\left(\frac{\partial U}{\partial T}\right)_{g,h} \geq 0 \tag{B.32}$$

is never negative. Here the remaining g and h are fixed. In fact, throughout the following argument $\{g_a\}$ and $\{h^b\}$ will be given and fixed; the only macroscopic quantity which is allowed to vary is the energy U. We assume that the energy is bounded from below and, moreover, that there exists a unique, non-degenerate ground state with energy $U_0 = U_0(g_a, h^b)$. It must be possible — at least in principle — to actually prepare the system in this ground state. The ground state is a pure state, and therefore $U = U_0$ implies $S = 0$. We further assume that the energy spectrum is not trivial, i.e. that there exist other energy levels $\epsilon_i \neq U_0$. But in any pure energy eigenstate the relative probabilities $\mathrm{Prob}(\epsilon_i)/\mathrm{Prob}(U_0) = \exp(-(\epsilon_i - U_0)/kT)$ must vanish, which is only possible if $T = 0$. Thus $S = 0$ implies $T = 0$. Since the heat capacity (B.32) is non-negative, the temperature of an arbitrary state (with $U \geq U_0$) may never be below the ground state temperature; thus

$$T \geq 0 \quad . \tag{B.33}$$

Conversely, $T = 0$ implies $U = U_0$. Hence we have established the equivalences

$$T = 0 \quad \Leftrightarrow \quad U = U_0 \quad \Leftrightarrow \quad S = 0 \quad ; \tag{B.34}$$

this is the 3rd law of thermodynamics.

Appendix C

Bosons and Fermions

C.1 Second Quantisation

Second quantisation entails the introduction of annihilation and creation operators. The operator which annihilates a particle in the state $|\psi\rangle$ is denoted by $a(\psi)$, and the operator which creates a particle in some other state $|\chi\rangle$ is denoted by $a^\dagger(\chi)$. These field operators are antilinear and linear, respectively, in the single-particle wave function:

$$\begin{aligned} a(\alpha_1\psi_1 + \alpha_2\psi_2) &= \alpha_1^* a(\psi_1) + \alpha_2^* a(\psi_2) \quad , \\ a^\dagger(\alpha_1\chi_1 + \alpha_2\chi_2) &= \alpha_1 a^\dagger(\chi_1) + \alpha_2 a^\dagger(\chi_2) \quad . \end{aligned} \tag{C.1}$$

The field operators satisfy commutation or anticommutation relations:

$$[a(\psi), a(\chi)]_\mp = [a^\dagger(\psi), a^\dagger(\chi)]_\mp = 0 \quad , \tag{C.2}$$

$$[a(\psi), a^\dagger(\chi)]_\mp = \langle\psi|\chi\rangle \quad , \tag{C.3}$$

depending on whether they describe bosons (upper sign) or fermions (lower sign). Field operators associated with some orthonormal basis $\{|i\rangle\}$ of single-particle states are also denoted by a_i, $a_j^\dagger$ for short. The right-hand side of the last (anti-)commutation relation then reduces to $\langle i|j\rangle = \delta_{ij}$.

C.2 Generalised Wick Theorem

One often faces the task of calculating expectation values of products of several field operators,

$$\langle ABC\ldots F\rangle \equiv \mathrm{tr}\,(\rho ABC\ldots F) \quad . \tag{C.4}$$

Here ρ denotes the statistical operator, and $A, B, \ldots$ represent arbitrary annihilation or creation operators. There is no general prescription for evaluating such expectation values unless ρ has the special *Hartree form*

$$\rho = Z^{-1} \exp\left(-\sum_{ij} \lambda^{ij} a_i^\dagger a_j\right) \quad . \tag{C.5}$$

In other words, the statistical operator has a canonical form with all $\{G_a\} = \{a_i^\dagger a_j\}$ being single-particle observables. In order for the statistical operator to be Hermitian, the matrix $\lambda \equiv (\lambda^{ij})$ of Lagrange parameters must be Hermitian:

$$\lambda^\dagger = \lambda \quad . \tag{C.6}$$

Whenever the statistical operator has the Hartree form, the generalised Wick theorem may be applied [101], yielding

$$\langle ABC \dots F \rangle = [A^\bullet B^\bullet C^{\bullet\bullet} \dots F^{\bullet\bullet\bullet}] + [A^\bullet B^{\bullet\bullet} C^\bullet \dots F^{\bullet\bullet\bullet}] + \dots \tag{C.7}$$

Here we have employed several shorthand notations. The 'bullets' $\bullet$ indicate *contractions*, defined by

$$A^\bullet B^\bullet := \mathrm{tr}\,(\rho AB) \quad . \tag{C.8}$$

Furthermore, it is understood that, e.g.,

$$[A^\bullet B^{\bullet\bullet} C^\bullet \dots F^{\bullet\bullet\bullet}] = \pm [A^\bullet C^\bullet B^{\bullet\bullet} \dots F^{\bullet\bullet\bullet}] \quad , \tag{C.9}$$

the sign depending on whether the particles under consideration are bosons or fermions. More generally, each rearrangement r of the field operators must be accompanied by a factor

$$\varepsilon(r) := (\pm 1)^{\deg(r)} \tag{C.10}$$

which is determined by the degree of the permutation involved. Note that not all permutations are permitted: the order of two operators which belong to the same contraction must be maintained. Wick's theorem can then be written in the symbolic form

$$\langle \prod \dots \rangle = \sum_r \varepsilon(r) \prod \text{contractions} \quad . \tag{C.11}$$

In order to evaluate each individual contraction, it is necessary to calculate the partition function

$$Z = \mathrm{tr}\ \exp\left(- \sum_{ij} \lambda^{ij} a_i^\dagger a_j \right) \quad . \tag{C.12}$$

One finds

$$Z = \det\left(1 \mp \mathrm{e}^{-\lambda}\right)^{\mp 1} \tag{C.13}$$

[114, 115], which in turn implies

$$\begin{aligned} \langle a_i^\dagger a_j \rangle &= -\frac{\partial}{\partial \lambda^{ij}} \ln Z \\ &= \left(\frac{1}{\mathrm{e}^\lambda \mp 1} \right)_{ji} \quad . \end{aligned} \tag{C.14}$$

The last step requires use of the identity

$$\ln \det = \mathrm{tr}\ \ln \quad . \tag{C.15}$$

C.3 Entropy

Let a_{ij} be a shorthand for $\langle a_i^\dagger a_j \rangle$. Then, in a symbolic matrix notation,

$$a^t = (\mathrm{e}^\lambda \mp 1)^{-1} \tag{C.16}$$

(with t denoting the transpose). Inversion yields

$$\lambda = \ln(a^{-1t} \pm 1) \quad . \tag{C.17}$$

The partition function may thus be expressed in terms of the expectation values (a_{ij}):

$$Z = [\det(1 \pm a^t)]^{\pm 1} \quad . \tag{C.18}$$

Since the transposition (t) does not affect the value of the determinant, it can be omitted. One finds

$$\ln Z = \pm \mathrm{tr}\ \ln(1 \pm a) \quad . \tag{C.19}$$

According to (B.8), the entropy is given by

$$\begin{aligned} S &= k \ln Z + k \sum_{ij} \lambda^{ij} a_{ij} \\ &= k \ln Z + k\, \mathrm{tr}\,(\lambda a^t) \quad . \end{aligned} \tag{C.20}$$

Using the above expressions for the partition function and the Lagrange parameters, one then obtains

$$S = -k\, \mathrm{tr}\,[a \ln a \mp (1 \pm a) \ln(1 \pm a)] \quad . \tag{C.21}$$

This result is valid for any state that has the Hartree form. In the special case when

$$a_{ij} = n_i \cdot \delta_{ij} \quad , \tag{C.22}$$

i.e., when all relevant observables are occupation numbers, then the formula for the entropy reduces to

$$S = -k \sum_i [n_i \ln n_i \mp (1 \pm n_i) \ln(1 \pm n_i)] \quad . \tag{C.23}$$

C.4 Relativistic Ideal Quantum Gases

Let

$$\sigma := \begin{cases} +1: & \text{bosons} \\ -1: & \text{fermions.} \end{cases} \tag{C.24}$$

Let N_j denote the number operator for the j-th energy level. The operator for the total particle number is then given by $N = \sum N_j$. We assume that the total energy of the system is the sum of one-particle energies, i.e., $H = \sum \epsilon_j N_j$. This yields the grand partition function

$$Z = \prod_j \left[1 - \sigma \mathrm{e}^{-\beta(\epsilon_j - \mu)}\right]^{-\sigma} \tag{C.25}$$

$$\ln Z = -\sigma \sum_j \ln \left[1 - \sigma \mathrm{e}^{-\beta(\epsilon_j - \mu)}\right] \quad . \tag{C.26}$$

From this we obtain the occupation number

$$\begin{aligned} n(\epsilon_i) &= -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_i} \ln Z \\ &= \left[\mathrm{e}^{\beta(\epsilon_i - \mu)} - \sigma\right]^{-1} \quad . \end{aligned} \tag{C.27}$$

For an ideal gas, $\ln Z$ can be calculated by replacing

$$\sum_j \to d\frac{V}{\hbar^3}\int \mathrm{d}^3 p \to \int_{mc^2}^{\infty} \mathrm{d}E\, g(E) \quad ; \tag{C.28}$$

here d is the degeneracy of the momentum eigenstates and $g(E)$ is the relativistic density of states

$$g(E) = \frac{dV}{2\pi^2(\hbar c)^3} E\sqrt{E^2 - m^2c^4} \quad . \tag{C.29}$$

By partial integration one obtains

$$\ln Z = \frac{dV}{6\pi^2(\hbar c\beta)^3} \int_0^{\infty} \mathrm{d}x\, \frac{(x^2 + 2\beta m x)^{3/2}}{\mathrm{e}^{-\beta(\mu - m)}\mathrm{e}^{x} - \sigma} \quad . \tag{C.30}$$

(In the integrand I have set $c = 1$.)

From the grand partition function one obtains the grand thermodynamic potential

$$\Omega(T, \mu, V) = -\frac{1}{\beta} \ln Z \quad , \tag{C.31}$$

which is defined as

$$\Omega := U - TS - \mu N \quad . \tag{C.32}$$

Since the system is homogeneous, one also has

$$\Omega = -pV \quad . \tag{C.33}$$

The partition function can be evaluated analytically in some special cases. We give two examples:

- $m = \mu = 0$. Then

$$\ln Z = \left(\frac{7}{8}\right)_f \frac{\pi^2}{90} \frac{dV}{(\hbar c\beta)^3} \quad . \tag{C.34}$$

The subscript f indicates that the prefactor only appears in the fermion case ($\sigma = -1$). According to (B.6), the energy of the system is

$$\begin{aligned} U &= -\left(\frac{\partial}{\partial\beta}\ln Z\right)_V \\ &= -3\Omega \\ &= 3pV \quad . \end{aligned} \tag{C.35}$$

- For a very dilute gas, $N/V \to 0$, the chemical potential diverges; more specifically, $-\beta(\mu - m) \to \infty$. The particle statistics ($\sigma = \pm 1$) then becomes irrelevant, and we obtain the *classical limit*

$$\ln Z = \frac{dV}{6\pi^2(\hbar c\beta)^3} \mathrm{e}^{\beta(\mu-m)} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}(x^2 + 2\beta m x)^{3/2} \quad . \tag{C.36}$$

According to (B.6), the particle number is given by

$$\begin{aligned} N &= -\left(\frac{\partial}{\partial(-\beta\mu)}\ln Z\right)_{\beta,V} \\ &= \ln Z \quad , \end{aligned} \tag{C.37}$$

from which we immediately obtain the ideal gas law:

$$pV = NkT \quad . \tag{C.38}$$

Bibliography

[1] B. Müller and A. Trayanov. Deterministic chaos in non-Abelian lattice gauge theory. *Phys. Rev. Lett.*, 68:3387, 1992.

[2] C. Gong. Lyapunov exponent of classical $su(3)$ gauge theory. *Phys. Lett. B*, 298:257, 1993.

[3] A. Casher, H. Neuberger, and S. Nussinov. Chromoelectric-flux-tube model of particle production. *Phys. Rev. D*, 20:179, 1979.

[4] N. K. Glendenning and T. Matsui. Creation of $q\overline{q}$ pairs in a chromoelectric flux tube. *Phys. Rev. D*, 28:2890, 1983.

[5] B. Banerjee, N. K. Glendenning, and T. Matsui. Meson radiation from a quark-gluon plasma by fission of chromoelectric flux tubes. *Phys. Lett. B*, 127:453, 1983.

[6] K. Sailer, T. Schönfeld, Z. Schram, A. Schäfer, and W. Greiner. Dynamical string model on the basis of a semiclassical unified string-flux tube model. *Int. J. Mod. Phys. A*, 6:4395, 1991.

[7] S. W. Hawking. Black hole explosions? *Nature*, 248:30, 1974.

[8] S. W. Hawking. Particle creation by black holes. *Commun. Math. Phys.*, 43:199, 1975.

[9] N. D. Birrell and P. C. W. Davies. *Quantum fields in curved space.* Cambridge University Press, Cambridge, 1984.

[10] T. Damour and R. Ruffini. Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism. *Phys. Rev. D*, 14:332, 1976.

[11] B. Müller, W. Greiner, and J. Rafelski. Interpretation of external fields as temperature. *Phys. Lett. A*, 63:181, 1977.

[12] C. R. Stephens. The Hawking effect in Abelian gauge theories. *Ann. Phys.*, 193:255, 1989.

[13] R. Brout, R. Parentani, and P. Spindel. Thermal properties of pairs produced by an electric field: a tunneling approach. *Nucl. Phys. B*, 353:209, 1991.

[14] P. Danielewicz. Quantum theory of nonequilibrium processes, I. *Ann. Phys.*, 152:239, 1984.

[15] C. H. Greiner. PhD thesis, Universität Erlangen, 1992. (unpublished).

[16] W. H. Zurek. Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? *Phys. Rev. D*, 24:1516, 1981.

[17] W. H. Zurek. Environment-induced superselection rules. *Phys. Rev. D*, 26:1862, 1982.

[18] J. J. Halliwell. Decoherence in quantum cosmology. *Phys. Rev. D*, 39:2912, 1989.

[19] A. Albrecht. Investigating decoherence in a simple system. *Phys. Rev. D*, 46:5504, 1992.

[20] B. L. Hu, J. P. Paz, and Y. Zhang. Quantum Brownian motion in a general environment: Exact master equation with nonlocal dissipation and colored noise. *Phys. Rev. D*, 45:2843, 1992.

[21] J. B. Hartle. Spacetime coarse grainings in nonrelativistic quantum mechanics. *Phys. Rev. D*, 44:3171, 1991.

[22] M. Gell-Mann and J. B. Hartle. Classical equations for quantum systems. *Phys. Rev. D*, 47:3345, 1993.

[23] W. H. Zurek. Decoherence and the transition from quantum to classical. *Physics Today*, page 36, October 1991.

[24] Negotiating the tricky border between quantum and classical. *Physics Today*, page 13, April 1993.

[25] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola. Pair production in a strong electric field. *Phys. Rev. Lett.*, 67:2427, 1991.

[26] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola. Fermion pair production in a strong electric field. *Phys. Rev. D*, 45:4659, 1992.

[27] F. Cooper, J. M. Eisenberg, Y. Kluger, E. Mottola, and B. Svetitsky. Particle production in the central rapidity region, 1992. preprint LA-UR-92-3856.

[28] F. Sauter. Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs. *Z. Phys.*, 69:742, 1931.

[29] F. Sauter. Zum "Kleinschen Paradoxon". *Z. Phys.*, 73:547, 1931.

[30] W. Heisenberg and H. Euler. Folgerungen aus der Diracschen Theorie des Positrons. *Z. Phys.*, 98:714, 1936.

[31] J. Schwinger. On gauge invariance and vacuum polarization. *Phys. Rev.*, 82:664, 1951.

[32] C. Zener. A theory of the electrical breakdown of solid dielectrics. *Proc. Roy. Soc. A*, 145:523, 1934.

[33] K. B. McAfee, E. J. Ryder, W. Shockley, and M. Sparks. Observations of Zener current in Germanium p-n junctions. *Phys. Rev.*, 83:650, 1951.

[34] L. V. Keldysh. Behavior of non-metallic crystals in strong electric fields. *Sov. Phys. JETP*, 6:763, 1958.

[35] L. V. Keldysh. The effect of a strong electric field on the optical properties of insulating crystals. *Sov. Phys. JETP*, 7:788, 1958.

[36] L. V. Keldysh. Ionization in the field of a strong electromagnetic wave. *Sov. Phys. JETP*, 20:1307, 1965.

[37] Y. A. Bychkov and A. M. Dykhne. Breakdown in semiconductors in an alternating electric field. *Sov. Phys. JETP*, 31:928, 1970.

[38] S. W. Hawking. Breakdown of predictability in gravitational collapse. *Phys. Rev. D*, 14:2460, 1976.

[39] S. W. Hawking. The unpredictability of quantum gravity. *Commun. Math. Phys.*, 87:395, 1982.

[40] C. Carlitz and R. Willey. Lifetime of a black hole. *Phys. Rev. D*, 36:2336, 1987.

[41] Y. Aharonov, A. Casher, and S. Nussinov. The unitarity puzzle and Planck mass stable particles. *Phys. Lett. B*, 191:51, 1987.

[42] D. N. Page. Is black-hole evaporation predictable? *Phys. Rev. Lett.*, 44:301, 1980.

[43] G. 't Hooft. On the quantum structure of a black hole. *Nucl. Phys. B*, 256:727, 1985.

[44] J. D. Bekenstein. How fast does information leak out from a black hole? *Phys. Rev. Lett.*, 70:3680, 1993.

[45] B. Müller. *The Physics of the Quark-Gluon Plasma.* Springer, Berlin, 1985. (Lecture Notes in Physics, Vol. 221).

[46] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert. *Relativistic Kinetic Theory.* North-Holland, Amsterdam, 1980.

[47] P. Carruthers and F. Zachariasen. Relativistic quantum transport theory approach to multiparticle production. *Phys. Rev. D*, 13:950, 1976.

[48] J. M. Eisenberg and G. Kälbermann. Pair production in transport equations. *Phys. Rev. D*, 37:1197, 1988.

[49] I. Bialynicki-Birula, P. Gornicki, and J. Rafelski. Phase-space structure of the Dirac vacuum. *Phys. Rev. D*, 44:1825, 1991.

[50] I. Prigogine. *Non-Equilibrium Statistical Mechanics.* Interscience, New York, London, 1962.

[51] R. Balian. *From Microphysics to Macrophysics, Vol. I.* Springer, Berlin, Heidelberg, 1991.

[52] E. T. Jaynes. Information theory and statistical mechanics. *Phys. Rev.*, 106:620, 1957.

[53] E. T. Jaynes. Information theory and statistical mechanics. II. *Phys. Rev.*, 108:171, 1957.

[54] G. Lindblad. On the generators of quantum dynamical semigroups. *Commun. Math. Phys.*, 48:119, 1976.

[55] W. Thirring. *A Course in Mathematical Physics 4: Quantum Mechanics of Large Systems.* Springer, New York, Wien, 1983.

[56] E. T. Jaynes. Information theory and statistical mechanics. In K. W. Ford, editor, *Statistical Physics.* Benjamin, 1963. (1962 Brandeis Theoretical Physics Lectures).

[57] E. T. Jaynes. *Papers on Probability, Statistics and Statistical Physics.* Kluwer Academic, Dordrecht, 1989. ed. by R. D. Rosenkrantz.

[58] A. Katz. *Principles of Statistical Mechanics.* Freeman, San Francisco, London, 1967.

[59] E. Fick and G. Sauermann. *The Quantum Statistics of Dynamic Processes.* Springer, Berlin, Heidelberg, 1990.

[60] R. Balian and N. L. Balazs. Equiprobability, inference, and entropy in quantum theory. *Ann. Phys.*, 179:97, 1987.

[61] R. Balian. *From Microphysics to Macrophysics, Vol. II.* Springer, Berlin, Heidelberg, 1992.

[62] E. T. Jaynes. Gibbs vs Boltzmann entropies. *Am. J. Phys.*, 33:391, 1965.

[63] S. Nakajima. On quantum theory of transport phenomena. *Prog. Theor. Phys.*, 20:948, 1958.

[64] R. Zwanzig. Ensemble method in the theory of irreversibility. *J. Chem. Phys.*, 33:1338, 1960.

[65] R. Zwanzig. Memory effects in irreversible thermodynamics. *Phys. Rev.*, 124:983, 1961.

[66] R. Zwanzig. Statistical mechanics of irreversibility. In W. E. Brittin, B. W. Downs, and J. Downs, editors, *Boulder Lectures in Theoretical Physics, Vol. 3*. Interscience, 1961.

[67] R. Zwanzig. On the identity of three generalized Master equations. *Physica (Utrecht)*, 30:1109, 1964.

[68] R. Zwanzig. Nonlinear generalized Langevin equations. *J. Stat. Phys.*, 9:215, 1973.

[69] H. Mori. Transport, collective motion, and Brownian motion. *Prog. Theor. Phys.*, 33:423, 1965.

[70] H. Mori. A continued-fraction representation of the time-correlation functions. *Prog. Theor. Phys.*, 34:399, 1965.

[71] B. Robertson. Equations of motion in nonequilibrium statistical mechanics. *Phys. Rev.*, 144:151, 1966.

[72] B. Robertson. Equations of motion of nuclear magnetism. *Phys. Rev.*, 153:391, 1967.

[73] B. Robertson. Equations of motion in nonequilibrium statistical mechanics. II. energy transport. *Phys. Rev.*, 160:175, 1967. (Erratum: 166:206, 1968).

[74] B. Robertson. Quantum statistical mechanical derivation of generalized hydrodynamic equations. *J. Math. Phys.*, 11:2482, 1970.

[75] B. Robertson. In R. D. Levine and M. Tribus, editors, *The Maximum Entropy Formalism.* M.I.T. Press, 1978.

[76] H. Grabert. *Projection Operator Techniques in Nonequilibrium Statistical Mechanics.* Springer, Berlin, Heidelberg, 1982.

[77] D. N. Zubarev and M. V. Tokarchuk. Nonequilibrium thermofield dynamics and the nonequilibrium statistical operator method. *Theor. Math. Phys.*, 88:876, 1991.

[78] T. Petrosky and I. Prigogine. Alternative formulation of classical and quantum dynamics for non-integrable systems. *Physica A*, 175:146, 1991.

[79] C. Cohen-Tannoudji, B. Diu, and F. Laloe. *Quantum Mechanics Vol. 2.* Wiley, New York, London, 1977.

[80] J. Reinhardt, B. Müller, and W. Greiner. Theory of positron production in heavy-ion collisions. *Phys. Rev. A*, 24:103, 1981.

[81] L. Van Hove. Quantum-mechanical perturbations giving rise to a statistical transport equation. *Physica (Utrecht)*, 21:517, 1955.

[82] L. Van Hove. The approach to equilibrium in quantum statistics. *Physica (Utrecht)*, 23:441, 1957.

[83] R. Kubo, M. Toda, and N. Hashitsume. *Statistical Physics II.* Springer, Berlin, Heidelberg, 1985.

[84] K. Kawasaki and J. D. Gunton. Theory of nonlinear transport processes: Nonlinear shear viscosity and normal stress effects. *Phys. Rev. A*, 8:2048, 1973.

[85] R. D. Richtmyer. *Principles of Advanced Mathematical Physics, Vol. I.* Springer, New York, 1978.

[86] M. D. Scadron. *Advanced Quantum Theory.* Springer, Berlin, Heidelberg, 1991. 2nd ed.

[87] P. Carruthers and F. Zachariasen. Quantum collision theory with phase-space distributions. *Rev. Mod. Phys.*, 55:245, 1983.

[88] A. I. Nikishov. Barrier scattering in field theory. removal of Klein paradox. *Nucl. Phys. B*, 21:346, 1970.

[89] A. I. Nikishov. Pair production by a constant external field. *Sov. Phys. JETP*, 30:660, 1970.

[90] E. Brezin and C. Itzykson. Pair production in vacuum by an alternating field. *Phys. Rev. D*, 2:1191, 1970.

[91] M. S. Marinov and V. S. Popov. Pair production in an electromagnetic field (case of arbitrary spin). *Sov. J. Nucl. Phys.*, 15:702, 1972.

[92] V. S. Popov. Pair production in a variable and homogeneous field as an oscillator problem. *Sov. Phys. JETP*, 35:659, 1972.

[93] V. S. Popov. The imaginary time method for periodic fields. *Sov. J. Nucl. Phys.*, 19:584, 1974.

[94] C. Itzykson and J.-B. Zuber. *Quantum Field Theory.* McGraw-Hill, New York, 1980.

[95] W. Greiner, B. Müller, and J. Rafelski. *Quantum Electrodynamics of Strong Fields.* Springer, Berlin, Heidelberg, 1985.

[96] A. M. Perelomov. Some remarks on fermion pair creation in an alternating external field. *Phys. Lett. A*, 39:353, 1972.

[97] A. M. Perelomov. Creation of fermion pairs in a homogeneous alternating external field. *Theor. Math. Phys.*, 19:368, 1974.

[98] M. S. Marinov and V. S. Popov. Electron-positron pair creation from vacuum induced by variable electric field. *Fortsch. Phys.*, 25:373, 1977.

[99] M. Born and V. Fock. Beweis des Adiabatensatzes. *Z. Phys.*, 51:165, 1928.

[100] W. Thirring. *A Course in Mathematical Physics 3: Quantum Mechanics of Atoms and Molecules.* Springer, New York, Wien, 1981.

[101] A. L. Fetter and J. D. Walecka. *Quantum Theory of Many-Particle Systems.* McGraw-Hill, 1971.

[102] A. M. Perelomov. Some remarks on boson pair creation in alternating external field. *Phys. Lett. A*, 39:165, 1972.

[103] C. Zener. Non-adiabatic crossing of energy levels. *Proc. Roy. Soc. A*, 137:696, 1932.

[104] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics.* Pergamon, Oxford, 1965. 2nd ed.

[105] A. M. Dykhne. Adiabatic perturbation of discrete spectrum states. *Sov. Phys. JETP*, 14:941, 1962.

[106] A. Joye, H. Kunz, and C.-E. Pfister. Exponential decay and geometric aspect of transition probabilities in the adiabatic limit. *Ann. Phys.*, 208:299, 1991.

[107] G. A. Hagedorn. Proof of the Landau-Zener formula in an adiabatic limit with small eigenvalue gaps. *Commun. Math. Phys.*, 136:433, 1991.

[108] M. Herrmann and J. Knoll. Space-time effects in the decay of strings by pair creation. *Phys. Lett. B*, 234:437, 1990.

[109] T. Schönfeld, A. Schäfer, B. Müller, K. Sailer, J. Reinhardt, and W. Greiner. Pair creation in a flux tube. *Phys. Lett. B*, 247:5, 1990.

[110] K. Sailer, Z. Hornyak, A. Schäfer, and W. Greiner. $q\overline{q}$ pair production in expanding hadronic flux tubes. *Phys. Lett. B*, 287:349, 1992.

[111] L. F. Abbott, E. Farhi, and M. B. Wise. Particle production in the new inflationary cosmology. *Phys. Lett. B*, 117:29, 1982.

[112] J. D. Jackson. *Classical Electrodynamics.* John Wiley, New York, 1975. 2nd ed.

[113] F. Cooper and E. Mottola. Quantum back reaction in scalar QED as an initial-value problem. *Phys. Rev. D*, 40:456, 1989.

[114] F. Reif. *Fundamentals of Statistical and Thermal Physics.* McGraw-Hill, 1965.

[115] W. Greiner, L. Neise, and H. Stöcker. *Theoretische Physik 9: Thermodynamik und Statistische Mechanik.* Verlag Harri Deutsch, Frankfurt, 1987.

VITA

Jochen Rau

21 May 1968	born in Frankfurt am Main, Germany
Aug. 1974 – July 1978	Viktoriaschule Kronberg i.Ts.
Sept. 1978 – June 1985	Altkönigschule Kronberg i.Ts. *Abitur*
Oct. 1985 – July 1989	Johann Wolfgang Goethe-Universität Frankfurt am Main, Germany: Study at the Institute for Theoretical Physics, diploma thesis on the production of supersymmetric particles in ultrarelativistic heavy-ion collisions (advisor: Prof. W. Greiner). *Diplom-Physiker*
Oct. 1989 – June 1990	University of Cambridge, England: Member of Trinity College and study at the Department of Applied Mathematics and Theoretical Physics (Part III of the Mathematical Tripos). *Certificate of Advanced Study in Mathematics*
Aug. 1990 – July 1993	Duke University, North Carolina, USA: Graduate study and Teaching/Research Assistant at the Department of Physics (advisor: Prof. B. Müller). *Doctor of Philosophy (Ph.D.)*